TRUNCATED MOMENT PROBLEMS IN THE CLASS OF GENERALIZED NEVANLINNA FUNCTIONS

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Abstract. Truncated moment problems in the class of generalized Nevanlinna functions are investigated. General solvability criteria will be established, covering both the even and odd problems, including complete parametrizations of solutions. The main new results concern the case where the corresponding Hankel matrix of moments is degenerate. One of the new effects which reveals in the indefinite case is that the degenerated moment problem may have infinitely many solutions. However, with a careful application of an indefinite analogue of a step-by-step Schur algorithm a complete description of the set of solutions will be obtained.

1. Introduction

The main purpose of this paper is to study general truncated (real) moment problems and some associated interpolation problems involving a finite sequence of real numbers $s_0, s_1, \dots, s_\ell$. In order to describe some of the contents and results in the paper it is natural to start by recalling a couple of notions and results appearing in classical truncated moment problems.

The truncated Hamburger moment problem for real numbers $s_0, s_1, \dots, s_\ell$ ($\ell \in \mathbb{Z}_+$) consists of finding a positive measure μ on $I = \mathbb{R}$ for which

$$\int_I t^j \, d\mu(t) = s_j, \quad j = 0, 1, \dots, \ell. \tag{1.1}$$

This problem will be called *odd* or *even* , if the number ℓ is odd or even, respectively. In the case where $I = \mathbb{R}_+$ the problem (1.1) is called the truncated Stieltjes moment problem. Due to the Hamburger-Nevanlinna theorem [1, Theorem 3.2.1] in the even case ($\ell = 2n$) the conditions in (1.1) can be rewritten in terms of the associated function $\varphi(\lambda)$ defined by

$$\varphi(\lambda) = \int_I \frac{d\mu(t)}{t-\lambda}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{1.2}$$

as the following interpolation problem at $\lambda = \infty$:

$$\varphi(\lambda) = -\frac{s_0}{\lambda} - \cdots - \frac{s_\ell}{\lambda^{\ell+1}} + o\left(\frac{1}{\lambda^{\ell+1}}\right), \quad \lambda \widehat{\to} \infty. \tag{1.3}$$

The notation $\lambda \widehat{\to} \infty$ means that $\lambda \to \infty$ nontangentially, i.e. $\delta < \arg \lambda < \pi - \delta$ for some $\delta > 0$. Recall, that φ belongs to the class $\mathbf{N}_0$ of Nevanlinna functions, i.e.,

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$\varphi(\lambda)$ is holomorphic on $\mathbb{C}\setminus\mathbb{R}$, satisfies the symmetry condition $\overline{\varphi(\lambda)}=\varphi(\bar\lambda)$, and has a nonnegative imaginary part for all $\lambda\in\mathbb{C}_+$. The moment problem (1.1) can now be reformulated as follows: find a Nevanlinna function $\varphi(\lambda)$ such that (1.3) holds. It follows easily from (1.1) that the following inequality

$$S_n := (s_{i+j})_{i,j=0}^n \geq 0, \tag{1.4}$$

is necessary for the problem (1.1) to be solvable. In the case where the matrix S_n is invertible this condition is also sufficient for (1.1) to be solvable, and all its solutions are described by the formula (see [28]):

$$\varphi(\lambda)=\int_{\mathbb{R}}\frac{d\mu(t)}{t-\lambda}=-\frac{Q_n(\lambda)\tau(\lambda)+Q_{n+1}(\lambda)}{P_n(\lambda)\tau(\lambda)+P_{n+1}(\lambda)}, \tag{1.5}$$

where P_n are polynomials of the first kind orthonormal with respect to $\mathfrak{S}$,

$$Q_n(\lambda)=\mathfrak{S}\left(\frac{P_n(t)-P_n(\lambda)}{t-\lambda}\right) \tag{1.6}$$

are polynomials of the second kind and $\tau(\lambda)$ is an arbitrary Nevanlinna function from the class $\mathbf{N_0}$ which satisfies the Nevanlinna condition

(E) $\qquad \tau(\lambda)=o(\lambda) \quad \text{as} \quad \lambda\widehat{\rightarrow}\infty;$

in (1.6) $\mathfrak{S}$ stands for the nonnegative functional defined on the set $\mathbb{C}[t]$ of polynomials via

$$\mathfrak{S}(t^j)=s_j,\quad j=0,1,\dots,2n.$$

Notice that this classical result depends essentially also on the assumption that the moment problem is even (i.e., $\ell=2n$). Indeed, the odd Hamburger moment problem is not equivalent to the interpolation problem (1.3). A convenient framework to formulate the problem in the odd case is provided by the classes $\mathbf{N}_{0,-\ell}$ appearing in [21]: they consist of functions $f\in\mathbf{N}_0$ of the form (1.2), such that the measure μ in (1.2) satisfies the condition

$$\int_{\mathbb{R}}(1+|t|^\ell)d\mu(t)<\infty \quad (\ell=-1,0,1,2,\dots).$$

Then the Hamburger-Nevanlinna theorem can be restated as follows: μ is a solution of the moment problem (1.1) if and only if the associated function φ belongs to the class $\mathbf{N}_{0,-\ell}$ and has the asymptotic expansion (1.3). It is a consequence of the results in the present paper (see Corollary 5.2) that the set of solutions of the nondegenerate odd moment problem (1.1) can also be given in the form (1.5), where τ now ranges over the class $\mathbf{N}_{0,1}$ and satisfies

(O) $\qquad \tau(\lambda)=o(1) \quad \text{as} \quad \lambda\widehat{\rightarrow}\infty.$

It should be mentioned that in the special case of the nondegenerate odd Stieltjes moment problem the set of solutions was parametrized in [23] with the parameter τ ranging over the class of Stieltjes functions (i.e. Nevanlinna functions of the form (1.2) with $I=\mathbb{R}_+$). Such functions automatically belong to the class $\mathbf{N}_{0,1}$ and they satisfy also the condition (O). However, it seems to the authors that for general measures, whose support is not contained in some semiaxis in $\mathbb{R}$, the above mentioned description of the solution set for nondegenerate odd Hamburger moment problem has not appeared in the literature earlier.

In the case where the matrix S_n is degenerate the condition (1.4) is not anymore sufficient for the problem (1.1) to be solvable; see [26, 22, 8]. Recall that the *Hankel rank*, denoted by rank $(\mathbf{s},2n)$, of the sequence $(\mathbf{s},2n)=\{s_j\}_{j=0}^{2n}$ is defined as follows:

rank $(\mathbf{s}, 2n) = n+1$ if $\det S_n \neq 0$, otherwise, rank $(\mathbf{s}, 2n)$ is the smallest integer r, $0 \le r \le n$, such that

$$\begin{pmatrix} s_r \\ \vdots \\ s_{r+n} \end{pmatrix} \in \operatorname{span} \left(\begin{pmatrix} s_0 \\ \vdots \\ s_n \end{pmatrix}, \dots, \begin{pmatrix} s_{r-1} \\ \vdots \\ s_{r-1+n} \end{pmatrix} \right). \tag{1.7}$$

By a Frobenius theorem (see [19, Lemma X.10.1]) Hankel rank of $(\mathbf{s}, 2n)$ is the smallest integer r, $1 \le r \le n+1$, such that

$$\det S_{r-1} \neq 0, \text{ and } \det S_j = 0 \text{ for } j \ge r.$$

In particular, rank $(\mathbf{s}, 2n) = 0$ if $s_0 = \cdots = s_n = 0$, otherwise rank $(\mathbf{s}, 2n)$ is the smallest integer r, $1 \le r \le n+1$, such that

$$\det S_{r-1} \neq 0, \text{ and } \det S_j = 0 \text{ for } j \ge r.$$

A sequence $(\mathbf{s}, 2n) = \{s_j\}_{j=0}^{2n}$ with the Hankel rank $r = \operatorname{rank}(\mathbf{s}, 2n)$ is called *recursively generated*, if there exist numbers $\alpha_0, \dots, \alpha_{r-1}$, such that

$$s_j = \alpha_0 s_{j-r} + \cdots + \alpha_{r-1} s_{j-1} \quad (r \le j \le 2n). \tag{1.8}$$

Now solvability criteria for the degenerate truncated moment (1.1) can be formulated as follows.

Theorem 1.1 ([8]). *Let the matrix $S_n = (s_{i+j})_{i,j=0}^n$ be nonnegative and degenerate, and let $r = \operatorname{rank} \mathbf{s}$. Then the following statements are equivalent:*

(i) *the moment problem* (1.1) *is solvable;*
(ii) $\operatorname{rank} S_n = r$*;*
(iii) S_n *admits a nonnegative Hankel extension* S_{n+1}*;*
(iv) *the sequence* $(\mathbf{s}, 2n) = \{s_j\}_{j=0}^{2n}$ *is recursively generated.*

If any of the assumptions (i)–(iv) are satisfied, then the problem (1.1) *has a unique solution* $\varphi(\lambda) = -\frac{Q_r(\lambda)}{P_r(\lambda)}$.

It is interesting to note that Theorem 1.1 contains as a corollary the following rigidity result due to D. Burns and S. Krantz [7]: if φ is a rational Nevanlinna function of degree r with the asymptotic expansion (1.3) and ψ is a Nevanlinna function such that $\psi(\lambda) = \varphi(\lambda) + o(\frac{1}{\lambda^{2r+1}})$ as $\lambda \widehat{\to} \infty$, then $\psi(\lambda) \equiv \varphi(\lambda)$.

The main subject of the present paper is the study of degenerate odd and even moment problems involving finite sequences $s_0, s_1, \dots, s_\ell$ of real numbers by means of functions belonging to the class of generalized Nevanlinna functions, which contains the class of Nevanlinna functions appearing in (1.2) as a subclass.

Definition 1.2. ([24]) Let $\kappa \in \mathbb{N}$. A function φ meromorphic on $\mathbb{C}_+$ is said to be from the class $\mathbf{N}_\kappa$, $\kappa \in \mathbb{N}$, of *generalized Nevanlinna functions* with κ negative squares, if the kernel

$$\mathsf{N}_\omega(\lambda) = \frac{\varphi(\lambda) - \overline{\varphi(\omega)}}{\lambda - \bar{\omega}}$$

has κ negative squares on $\mathbb{C}_+$, i.e. for every choice of $m \in \mathbb{N}$, $\lambda_1, \dots, \lambda_m \in \mathbb{C}_+$ the matrix

$$(\mathsf{N}_{\lambda_k}(\lambda_i))_{i,k=1}^m$$

has at most κ and for some choice of m, λ_j exactly κ negative eigenvalues.

In [12, 14] (see Definitions 2.3, 2.7 below) subclasses $\mathbf{N}_{\kappa,-\ell}$ of the class $\mathbf{N}_\kappa$ were introduced as indefinite analogues of the subclasses $\mathbf{N}_{0,-\ell}$ appearing in [21].

In this paper we consider in a parallel way the following two problems:

Indefinite truncated moment problem $MP_\kappa(s,\ell)$: Given are $\kappa\in\mathbb{N}$, $\ell\in\mathbb{Z}_+$, and $s_0,\dots,s_\ell\in\mathbb{R}$. Find a function $\varphi\in\mathbf{N}_{\kappa,-\ell}$ with the asymptotic expansion (1.3). Denote by $\mathcal{M}_\kappa(s,\ell)$ the set of solutions of this problem.

Multiple indefinite interpolation problem $IP_\kappa(s,\ell)$: Given are $\kappa\in\mathbb{N}$, $\ell\in\mathbb{Z}_+$, and $s_0,\dots,s_\ell\in\mathbb{R}$. Find a function $\varphi\in\mathbf{N}_\kappa$ with the asymptotic expansion (1.3). The set of functions with these properties is denoted by $\mathcal{I}_\kappa(s,\ell)$.

As was mentioned above, the problem $MP_0(s,\ell)$ is equivalent to the truncated Hamburger moment problem (1.1). Furthermore, in the even case ($\ell=2n$) $\mathcal{M}_\kappa(s,\ell)=\mathcal{I}_\kappa(s,\ell)$, while in the odd case ($\ell=2n+1$) we have $\mathcal{M}_\kappa(s,\ell)\subset\mathcal{I}_\kappa(s,\ell)$, but the reverse inclusion fails to hold in general.

The problems $MP_\kappa(s,\ell)$, $IP_\kappa(s,\ell)$ will be called nondegenerate, if

$$\det S_n\neq 0,\ \text{ for } n=[\ell/2]; \tag{1.9}$$

otherwise they are called degenerate. The indices j for which $\det S_{j-1}\neq 0$ are called *normal indices* of the Hankel matrix S_n. Let

$$n_1<n_2<\cdots<n_N\le n+1$$

be the sequence of all *normal indices* of the matrix S_n. In the case of arbitrary Hankel matrix S_n we show that the largest normal index n_N of S_n coincides with the Hankel rank of the sequence $\mathbf{s}$.

A necessary condition for the problems $MP_\kappa(\mathbf{s},\ell)$, $IP_\kappa(s,\ell)$ to be solvable is that

$$\kappa\ge\nu_-(S_n), \tag{1.10}$$

where $\nu_-(S_n)$ is the total multiplicity of all negative eigenvalues of S_n. The method we use for the solution of the moment problem $MP_\kappa(\mathbf{s},2n)$ and $IP_\kappa(s,\ell)$ for $\kappa\ge\nu_-(S_n)$ is based on the Schur-Chebyshev recursion algorithm, studied in the nondegenerate situation by M. Derevjagin [9] (see also [4]). With this method every solution φ of the moment problem $MP_\kappa(\mathbf{s},2n)$ can be obtained via

$$\varphi(\lambda)=\frac{-s_{n_1-1}}{p_1(\lambda)+\varepsilon_1\varphi_1(\lambda)}, \tag{1.11}$$

where $p_1(\lambda)=P_{n_1}(\lambda)$, $\varepsilon_1=\operatorname{sgn}s_{n_1-1}$, and φ_1 is a solution of an "induced" moment problem $MP_{\kappa-\kappa_1}(\mathbf{s}^{(1)},2(n-n_1))$ with $\kappa_1=\nu_-(S_{n_1-1})$.

In the case of a nondegenerate moment problem the condition (1.10) is also sufficient for the problem $MP_\kappa(\mathbf{s},\ell)$ to be solvable and subsequent applications of the formula (1.11) shows that $\mathcal{M}_\kappa(\mathbf{s},\ell)$ in the even case ($\ell=2n$) is parametrized via the linear fractional transformation

$$\varphi(\lambda)=-\frac{Q_{n_{N-1}}(\lambda)\tau(\lambda)+Q_{n_N}(\lambda)}{P_{n_{N-1}}(\lambda)\tau(\lambda)+P_{n_N}(\lambda)}, \tag{1.12}$$

with the parameter τ ranging over the class $\mathbf{N}_{\kappa-\nu_-(S_n)}$ and satisfying the Nevanlinna condition (E); see [18, 11]. In this formula P_j and Q_j are polynomials of the first and the second types introduced in [11]. In the odd case a similar description of the sets $\mathcal{M}_\kappa(\mathbf{s},\ell)$ and $\mathcal{I}_\kappa(\mathbf{s},\ell)$ is given in Theorem 5.1, with the parameter τ ranging over the class $\mathbf{N}_{\kappa-\nu_-(S_n),1}$ and $\mathbf{N}_{\kappa-\nu_-(S_n)}$, respectively, and satisfying the condition (O). It should be mentioned, that this result for $\mathcal{I}_\kappa(\mathbf{s},\ell)$ can be derived

also from the recent paper [5] on boundary interpolation in generalized Nevanlinna classes.

Now let us briefly describe some of the main results obtained in the present paper for the degenerate indefinite truncated moment problem in the even case. As in the definite case, for a degenerate problem the condition (1.10) is not sufficient for the problem $MP_\kappa(\mathbf{s}, 2n)$ to be solvable. The following theorem gives some solvability criteria in the special case, where $\kappa = \nu_-(S_n)$; in fact, this result offers a natural generalization for the results due to Curto and Fialkow [8], which were formulated in Theorem 1.1 above.

Theorem 1.3. *Let $n_1 < n_2 < \cdots < n_N$ be the sequence of all normal indices of a degenerate matrix $S_n = (s_{i+j})_{i,j=0}^n$ and let $\kappa = \nu_-(S_n)$. Then the following statements are equivalent:*

(i) *the moment problem $MP_\kappa(\mathbf{s}, 2n)$ is solvable;*
(ii) $\operatorname{rank} S_n = n_N$*;*
(iii) *S_n admits a Hankel extension S_{n+1} such that*

$$\nu_-(S_{n+1}) = \nu_-(S_n);$$

(iv) *the sequence $\mathbf{s} = \{s_j\}_{j=0}^{2n}$ is recursively generated.*

If any of the assumptions (i)–(iv) is satisfied, then the problem $MP_\kappa(\mathbf{s}, 2n)$ has a unique solution $\varphi(\lambda) = -\frac{Q_{n_N}(\lambda)}{P_{n_N}(\lambda)}$.

As a consequence of Theorem 1.3 a rigidity result for generalized Nevanlinna functions from [5] can be derived (see also [6]).

A new effect which appears in the indefinite case is that the degenerate moment problem $MP_\kappa(\mathbf{s}, 2n)$ has infinitely many solutions for κ large enough. As will be shown below, the problem $MP_\kappa(\mathbf{s}, 2n)$ with $\kappa > \nu_-(S_n)$ is solvable if and only if

$$\kappa \geq \nu_-(S_n) + \nu_0(S_n). \tag{1.13}$$

If, in addition, $\operatorname{rank} S_n = n_N + 1$ and ν satisfies some appropriate further conditions (see (2.17) below), then the solution set $\mathcal{M}_\kappa(\mathbf{s}, 2n)$ can be described by the formula (1.12), where

$$\tau(\lambda) = \frac{\widehat{\tau}(\lambda)}{\lambda^{2\nu_0}}, \tag{1.14}$$

and $\widehat{\tau}$ is a function from the class $\mathbf{N}_{\kappa-\nu}$, which satisfies (E); see Theorem 5.8.

On the other hand, if (1.13) holds and $\operatorname{rank} S_n > n_N + 1$, then the solution set $\mathcal{M}_\kappa(\mathbf{s}, 2n)$ is described by (1.12), where

$$\tau(\lambda) = \frac{-\varepsilon}{\lambda^{2\nu_0}(\widehat{p}(\lambda) + \varepsilon\widehat{\tau}(\lambda))}, \tag{1.15}$$

$\widehat{p}$ is a polynomial of degree $n - \nu_0 + 1$ (as given in (3.15) below), and $\widehat{\tau}$ is a function from the class $\mathbf{N}_{\kappa-\nu}$, which satisfies (E); see Theorem 5.9.

In the odd case the degenerate indefinite truncated moment problem can be treated analogously. A condition, similar to (1.13) appeared in [29] as a solvability condition for a degenerate indefinite Nevanlinna-Pick interpolation problem.

In Section 2 the basic tools needed in this paper are given. Solutions to the so-called basic moment and interpolation problems will be described in Section 3. Section 4 describes a general Schur-Chebyshev recursion algorithm, which makes use of the normal indices of the associated Hankel matrix $S_n = (s_{i+j})_{i,j=0}^n$ defined

in (1.4). In Section 5 solvability criteria and complete descriptions for the set of solutions of the problems $MP_\kappa(\mathbf{s},\ell)$ and $IP_\kappa(\mathbf{s},\ell)$ in the general setting are established. Finally, Appendix contains some results on block matrices, which are needed in this paper; however, they may be also of independent interest: for instance, Lemma A.2 gives an extension of a well-know result on nonnegative block matrices.

2. Preliminaries

2.1. **Canonical factorizations of generalized Nevanlinna functions.** The definition of the class $\mathbf{N}_\kappa$ of generalized Nevanlinna functions is given in the Introduction. Clearly, if $\varphi\in\mathbf{N}_\kappa$ and $\varphi\not\equiv 0$, then also $-1/\varphi\in\mathbf{N}_\kappa$.

Recall (see [25]), that the point $\alpha\in\mathbb{R}$ is called a *generalized pole of nonpositive type* (GPNT) of the function $\varphi\in\mathbf{N}_\kappa$ with multiplicity $\kappa_\alpha(\varphi)$ if

$$-\infty<\lim_{z\widehat{\to}\alpha}(z-\alpha)^{2\kappa_\alpha+1}\varphi(z)\le 0,\quad 0<\lim_{z\widehat{\to}\alpha}(z-\alpha)^{2\kappa_\alpha-1}\varphi(z)\le\infty. \tag{2.1}$$

Similarly, the point ∞ is called a *generalized pole of nonpositive type* of φ with multiplicity $\kappa_\infty(\varphi)$ if

$$0\le\lim_{z\widehat{\to}\infty}\frac{\varphi(z)}{z^{2\kappa_\infty+1}}<\infty,\quad -\infty\le\lim_{z\widehat{\to}\infty}\frac{\varphi(z)}{z^{2\kappa_\infty-1}}<0. \tag{2.2}$$

A point $\beta\in\mathbb{R}\cup\{\infty\}$ is called a *generalized zero of nonpositive type* (GZNT) of the function $\varphi\in\mathbf{N}_\kappa$ if β is a generalized pole of nonpositive type of the function $-1/\varphi$. The multiplicity $\pi_\beta(\varphi)$ of the generalized zero of nonpositive type β of φ can be characterized by the inequalities:

$$0<\lim_{z\widehat{\to}\beta}\frac{\varphi(z)}{(z-\beta)^{2\pi_\beta+1}}\le\infty,\quad -\infty<\lim_{z\widehat{\to}\beta}\frac{\varphi(z)}{(z-\beta)^{2\pi_\beta-1}}\le 0. \tag{2.3}$$

Similarly, the point ∞ is a *generalized zero of nonpositive type* of φ with multiplicity $\pi_\infty(\varphi)$ if

$$-\infty\le\lim_{z\widehat{\to}\infty}z^{2\pi_\infty+1}\varphi(z)<0,\quad 0\le\lim_{z\widehat{\to}\infty}z^{2\pi_\infty-1}\varphi(z)<\infty. \tag{2.4}$$

Remark 2.1. If $\varphi_1\in\mathbf{N}_{\kappa_1}$ and $\varphi_2\in\mathbf{N}_{\kappa_2}$ then $\varphi_1+\varphi_2$ belongs to $\mathbf{N}_\kappa$, where $\kappa\le\kappa_1+\kappa_2$. It was shown by M.G. Kreĭn and H. Langer in [25] for $\varphi\in\mathbf{N}_\kappa$ that the total multiplicity of poles (zeros) in $\mathbb{C}_+$ and generalized poles (zeros) of nonpositive type in $\mathbb{R}\cup\{\infty\}$ is equal to κ. As a corollary of this result one obtains that if $\varphi_1\in\mathbf{N}_{\kappa_1}$ and $\varphi_2\in\mathbf{N}_{\kappa_2}$ have no common poles in $\mathbb{C}_+$ and common generalized poles of nonpositive type in $\mathbb{R}\cup\{\infty\}$ then $\varphi_1+\varphi_2\in\mathbf{N}_{\kappa_1+\kappa_2}$.

The generalized poles and zeros of nonpositive type of a generalized Nevanlinna function give rise to the following factorization result ([17], see also [12]).

Theorem 2.2. *Let $\varphi\in\mathbf{N}_\kappa$ and let $\alpha_1,\dots,\alpha_l$ ($\beta_1,\dots,\beta_m$) be all the generalized poles (zeros) of nonpositive type of φ in $\mathbb{R}$ and the poles (zeros) of φ in $\mathbb{C}_+$ with multiplicities $\kappa_1,\dots,\kappa_l$ ($\pi_1,\dots,\pi_m$). Then the function φ admits a (unique) canonical factorization of the form*

$$\varphi(z)=r(z)r^\#(z)\varphi_0(z), \tag{2.5}$$

where $\varphi_0\in\mathbf{N}_0$, $r^\#(z)=\overline{r(\bar z)}$, and $r=p/q$ with relatively prime polynomials

$$p(z)=\prod_{j=1}^m(z-\beta_j)^{\pi_j},\quad q(z)=\prod_{j=1}^l(z-\alpha_j)^{\kappa_j},$$

of degree $\kappa - \pi_\infty(\varphi)$ and $\kappa - \kappa_\infty(\varphi)$, respectively.

It follows from (2.5) that the function φ admits the (factorized) integral representation

$$\varphi(z) = r(z)r^{\#}(z)\left(a + bz + \int_{\mathbb{R}}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)d\rho(t)\right), \quad r = \frac{p}{q}, \tag{2.6}$$

where $a \in \mathbb{R}$, $b \geq 0$, and $\rho(t)$ is a nondecreasing function satisfying the integrability condition

$$\int_{\mathbb{R}} \frac{d\rho(t)}{t^2+1} < \infty. \tag{2.7}$$

2.2. **The subclasses $\mathbf{N}_{\kappa,-\ell}$ of generalized Nevanlinna functions.**

Definition 2.3. (see [12]) A function $\varphi \in \mathbf{N}_\kappa$ is said to belong to the subclass $\mathbf{N}_{\kappa,1}$, if

$$\lim_{z \widehat{\to} \infty} \frac{\varphi(z)}{z} = 0 \text{ and } \int_\eta^\infty \frac{|\operatorname{Im}\varphi(iy)|}{y}\, dy < \infty, \tag{2.8}$$

with $\eta > 0$ large enough. Similarly, a function $\varphi \in \mathbf{N}_\kappa$ is said to belong to the subclass $\mathbf{N}_{\kappa,0}$, if

$$\lim_{z \widehat{\to} \infty} \frac{\varphi(z)}{z} = 0 \text{ and } \limsup_{z \widehat{\to} \infty} |z \operatorname{Im}\varphi(z)| < \infty. \tag{2.9}$$

Remark 2.4. Every function $\varphi \in \mathbf{N}_{\kappa,1}$ has a nontangential limit $\lim_{\lambda \widehat{\to} \infty} \varphi(\lambda)$ at infinity. As was shown in [12] the following implication holds:

$$\varphi \in \mathbf{N}_{\kappa,1}, \quad \lim_{\lambda \widehat{\to} \infty} \varphi(\lambda) \neq 0 \Rightarrow -1/\varphi \in \mathbf{N}_{\kappa,1}.$$

In the following theorem the subclasses $\mathbf{N}_{\kappa,1}$ and $\mathbf{N}_{\kappa,0}$ are characterized in terms of the integral representation (2.6).

Theorem 2.5. ([12]) *For $\varphi \in \mathbf{N}_\kappa$ and $\ell = 0, 1$ the following statements are equivalent:*

(i) *φ belongs to $\mathbf{N}_{\kappa,\ell}$;*

(ii) *φ has the integral representation* (2.6) *with* $\deg q - \deg p = \pi_\infty(\varphi) > 0$*, or with* $\deg p = \deg q$ *($\pi_\infty(\varphi) = 0$), $b = 0$, and*

$$\int_{\mathbb{R}} (1 + |t|)^{-\ell} d\rho(t) < \infty. \tag{2.10}$$

Remark 2.6. If $\varphi \in \mathbf{N}_{\kappa,0}$ then the statement (ii) in Theorem 2.5 can be strengthened in the sense that for every function $\varphi \in \mathbf{N}_{\kappa,0}$ there are real numbers γ and s_0, such that

$$\varphi(z) = \gamma - \frac{s_0}{z} + o\left(\frac{1}{z}\right), \quad z \widehat{\to} \infty. \tag{2.11}$$

Definition 2.7. ([14]) A function $\varphi \in \mathbf{N}_\kappa$ is said to belong to the subclass $\mathbf{N}_{\kappa,-2n}$, $n \in \mathbb{N}$, if there are real numbers γ and $s_0, \dots, s_{2n-1}$ such that the function

$$\varphi^{[2n]}(z) = z^{2n}\left(\varphi(z) - \gamma + \sum_{j=1}^{2n} \frac{s_{j-1}}{z^j}\right) \tag{2.12}$$

is $O(1/z)$ as $z \widehat{\to} \infty$. Moreover, $\varphi \in \mathbf{N}_\kappa$ is said to belong to the subclass $\mathbf{N}_{\kappa,-2n+1}$ if the function $\varphi^{[2n]}$ in (2.12) belongs to $\mathbf{N}_{\kappa',1}$ for some $\kappa' \in \mathbb{N}$, $\kappa' \leq \kappa$.

As was shown in [14], the following inclusions are satisfied

$$\cdots \subset \mathbf{N}_{\kappa,-2n-1} \subset \mathbf{N}_{\kappa,-2n} \subset \mathbf{N}_{\kappa,-2n+1} \subset \cdots \subset \mathbf{N}_{\kappa,0} \subset \mathbf{N}_{\kappa,1}. \tag{2.13}$$

The subclasses $\mathbf{N}_{\kappa,-\ell}$, $\ell \in \mathbb{N}$, can be characterized by means of the integral representation of φ in (2.6).

Theorem 2.8. *([14]) For $\varphi \in \mathbf{N}_\kappa$ the following statements are equivalent:*

(i) *$\varphi \in \mathbf{N}_{\kappa,-\ell}$, $\ell \in \mathbb{N}$;*
(ii) *φ has an integral representation (2.6) with $\pi_\infty(\varphi) = \deg q - \deg p \geq 0$ (and $b = 0$ if $\pi_\infty(\varphi) = 0$), such that*

$$\int_{\mathbb{R}} (1+|t|)^{\ell - 2\pi_\infty} d\rho(t) < \infty. \tag{2.14}$$

Remark 2.9. By Definition 2.7 every function $\varphi \in \mathbf{N}_{\kappa,-\ell}$ with odd ℓ admits the asymptotic expansion

$$\varphi(\lambda) = \gamma - \frac{s_0}{\lambda} - \cdots - \frac{s_\ell}{\lambda^{\ell+1}} + o\left(\frac{1}{\lambda^{\ell+1}}\right), \quad \lambda \widehat{\to} \infty. \tag{2.15}$$

If $\varphi \in \mathbf{N}_{\kappa,-\ell}$ and ℓ is even due to Theorem 2.8 there exists a real number s_{2n}, such that (2.15) holds. Conversely, if $\varphi \in \mathbf{N}_\kappa$ and satisfies (2.15) for even ℓ, then $\varphi \in \mathbf{N}_{\kappa,-\ell}$. This proves that in the even case ($\ell = 2n$) $\mathcal{M}_\kappa(s,\ell) = \mathcal{I}_\kappa(s,\ell)$, while in the odd case ($\ell = 2n+1$) the $\mathcal{M}_\kappa(s,\ell) \subset \mathcal{I}_\kappa(s,\ell)$.

The following Lemma is immediate from Definition 2.7, Theorem 2.8 and Remark 2.1.

Lemma 2.10. *Let $\varphi \in \mathbf{N}_{\kappa,-\ell}$ let $\nu_0 \leq \min\{\pi_\infty(\varphi), \ell/2\}$ and let*

$$\widehat{\varphi}(\lambda) = \lambda^{2\nu_0}\varphi(\lambda). \tag{2.16}$$

Then $\widehat{\varphi} \in \mathbf{N}_{\kappa-\nu,-(\ell-2\nu_0)}$, where

$$\nu = \begin{cases} \nu_0, & \text{if} \quad \kappa_0(\varphi) > 0; \\ \kappa_0(\varphi), & \text{if} \quad 0 \leq \kappa_0(\varphi) \leq \nu_0; \end{cases} \tag{2.17}$$

Conversely, if $\widehat{\varphi} \in \mathbf{N}_{\widehat{\kappa},-\widehat{\ell}}$, and $\kappa_\infty(\widehat{\varphi}) = 0$, φ and ν are given by (2.16) and (2.17), then $\varphi \in \mathbf{N}_{\widehat{\kappa}+\nu,-(\widehat{\ell}+2\nu_0)}$ and $\nu_0 \leq \pi_\infty(\varphi)$.

Proof. 1) Let $\varphi \in \mathbf{N}_{\kappa,-\ell}$. Due to Theorem 2.8 φ admits the canonical factorization (2.5), where the measure ρ in the integral representation (2.6) satisfies the condition (2.14). It follows from (2.5) and (2.17) that $\widehat{\varphi}$ admits the canonical factorization

$$\widehat{\varphi} = \frac{\widehat{p}(\lambda)\widehat{p}^{\#}(\lambda)}{\widehat{q}(\lambda)\widehat{q}^{\#}(\lambda)}\varphi_0(\lambda),$$

with the same function $\varphi_0 \in \mathbf{N}_0$ and

$$\pi_\infty(\widehat{\varphi}) = \deg \widehat{q} - \deg \widehat{p} = \pi_\infty(\varphi) - \nu_0.$$

Hence, the condition (2.14) takes the form

$$\int_{\mathbb{R}} (1+|t|)^{\widehat{\ell} - 2\pi_\infty(\widehat{\varphi})}, d\rho(t) < \infty. \tag{2.18}$$

where $\widehat{\ell} = \ell - 2\nu_0$. Due to Theorem 2.8 the latter condition is equivalent to the inclusion

$$\widehat{\varphi} \in \mathbf{N}_{\widehat{\kappa},-(\ell-2\nu_0)}$$

with $\widehat{\kappa} \le \kappa$.

2) Since $\nu_0 \le \pi_\infty(\varphi)$ then neither φ nor $\widehat{\varphi}$ has a GPNT at ∞. Assume that $\kappa_0(\varphi) > \nu_0$. Then both φ and $\widehat{\varphi}$ have GPNTs at 0 and $\kappa_0(\varphi) = \kappa_0(\widehat{\varphi}) + \nu_0$. Counting the total pole multiplicities of $\widehat{\varphi}$ one obtains $\widehat{\varphi} \in \mathbf{N}_{\kappa-\nu_0}$ by Remark 2.1.

Assume that $0 < \kappa_0(\varphi) \le \nu_0$. Then $\widehat{\varphi}$ has no GPNT at 0 and φ has a GPNT at 0 of multiplicity $\kappa_0(\varphi)$. Therefore, $\widehat{\varphi} \in \mathbf{N}_{\kappa-\kappa_0(\varphi)}$.

And finally, if $\kappa_0(\varphi) = 0$ and $\pi_0(\varphi) \ge 0$, then neither $\widehat{\varphi}$ nor φ have a GPNT at 0 and thus $\widehat{\varphi} \in \mathbf{N}_\kappa$. All the above statements are easily reversed. $\square$

2.3. **Toeplitz matrices.** A sequence $(\mathbf{c}, n) := (c_0, \dots, c_n)$ of (real or complex) numbers determines an upper triangular Toeplitz matrix $T(c_0, \dots, c_n)$ of order $(n+1) \times (n+1)$ with entries $t_{i,j} = c_{j-i}$ for $i \le j$ and $t_{i,j} = 0$ for $i > j$.

$$T(c_m, \dots, c_j) = \begin{pmatrix} c_m & \dots & c_j \\ & \ddots & \vdots \\ & & c_m \end{pmatrix}, \quad 0 \le m < j \le n. \tag{2.19}$$

Clearly, $T(c_m, \dots, c_j)^* = J_{j-m+1} T(\bar{c}_m, \dots, \bar{c}_j) J_{j-m+1}$, where

$$J_{j-m+1} = \begin{pmatrix} \mathbf{0} & & 1 \\ & \iddots & \\ 1 & & \mathbf{0} \end{pmatrix}. \tag{2.20}$$

In particular, the coefficients of the asymptotic expansions

$$\begin{aligned} c(\lambda) &= c_0 + \frac{c_1}{\lambda} + \dots + \frac{c_n}{\lambda^n} + o\left(\frac{1}{\lambda^n}\right), \quad \lambda \widehat{\to} \infty; \\ d(\lambda) &= d_0 + \frac{d_1}{\lambda} + \dots + \frac{d_n}{\lambda^n} + o\left(\frac{1}{\lambda^n}\right), \quad \lambda \widehat{\to} \infty. \end{aligned} \tag{2.21}$$

determine the Toeplitz matrices

$$T(c_0, \dots, c_n) = \begin{pmatrix} c_0 & \dots & c_n \\ & \ddots & \vdots \\ & & c_0 \end{pmatrix}, \; T(d_0, \dots, d_n) = \begin{pmatrix} d_0 & \dots & d_n \\ & \ddots & \vdots \\ & & d_0 \end{pmatrix}. \tag{2.22}$$

Lemma 2.11. *Let the functions c and d (meromorphic on $\mathbb{C} \setminus \mathbb{R}$) have the asymptotic expansions* (2.21) *and let $a(\lambda) = c(\lambda) d(\lambda)$ have the asymptotic expansion*

$$a(\lambda) = a_0 + \frac{a_1}{\lambda} + \dots + \frac{a_n}{\lambda^n} + o\left(\frac{1}{\lambda^n}\right), \quad \lambda \widehat{\to} \infty.$$

Then the first $n+1$ coefficients of the asymptotic expansion of $a(\lambda)$ can be found by

$$T(a_0, \dots, a_n) = T(c_0, \dots, c_n) T(d_0, \dots, d_n). \tag{2.23}$$

Lemma 2.12. *The formula*

$$p(\lambda) = \frac{1}{\det S_m} \det \begin{pmatrix} & & s_m & s_{m+1} \\ & \iddots & \iddots & \vdots \\ s_m & s_{m+1} & \dots & s_{2m+1} \\ 1 & \lambda & \dots & \lambda^{m+1} \end{pmatrix}, \tag{2.24}$$

where $s_j = 0, j < m$, $s_m \neq 0$ ($m \geq 0$), and S_m is as in (1.4)*, defines a monic polynomial $p(\lambda) = \sum_{j=0}^{m+1} p_j \lambda^j$ of degree $m+1$ whose coefficients p_j, $1 \leq j \leq m+1$, satisfy the matrix equality*

$$T(p_{m+1}, \dots, p_1)\, T(s_m, \dots, s_{2m}) = s_m I_{m+1} \quad (p_{m+1} = 1) \tag{2.25}$$

for an arbitrary real number s_{2m+1}.

Proof. Evaluating the determinant in (2.24) with respect to the last row shows immediately that $p(\lambda)$ is a monic polynomial of degree $m+1$. To see that the coefficients p_j of $p(\lambda)$ in (2.24) satisfy (2.25) substitute λ^j by s_{j+k-1} for $j = 0, 1, \dots, m+1$ ($s_{-1} = 0$) in the formula (2.24). Then for $k = 0$ the evaluation of the determinant in (2.24) yields the equality

$$\Sigma_{j=1}^{m+1} p_j s_{j-1} = s_m,$$

and for $k = 1, \dots, m$ one obtains

$$\Sigma_{j=1}^{m+1} p_j s_{j+k-1} = 0.$$

This means that the polynomial p defined by (2.24) automatically satisfies (2.25) for arbitrary $s_{2m+1} \in \mathbb{R}$. Note that s_{2m+1} only appears in the constant coefficient p_0 of $p(\lambda)$, which can be seen e.g. by evaluating the determinant in (2.24) with respect to the last column. □

2.4. **Asymptotic expansions of certain fractional transforms.** In the next lemma the polynomial $p(\lambda)$ defined in Lemma 2.12 (see (2.24)) appears when inverting an associated asymptotic expansion.

Lemma 2.13. *Let $(\mathbf{s}, \ell) = (s_j)_{j=0}^{\ell}$ be a sequence of real numbers such that $s_j = 0$, $j < m$, and $s_m \neq 0$, $\ell \geq 2m$, and let the monic polynomial $p(\lambda) = \sum_{j=0}^{m+1} p_j \lambda^j$ be defined by* (2.24)*. Then a function φ (meromorphic on $\mathbb{C}\backslash\mathbb{R}$) admits the asymptotic expansion*

$$\varphi(\lambda) = -\frac{s_m}{\lambda^{m+1}} - \cdots - \frac{s_\ell}{\lambda^{\ell+1}} + o\left(\frac{1}{\lambda^{\ell+1}}\right), \tag{2.26}$$

if and only if the function $-s_m/\varphi(\lambda)$ admits the asymptotic expansion

$$-s_m/\varphi(\lambda) = p(\lambda) + \varepsilon\tau(\lambda) \quad \lambda \widehat{\rightarrow} \infty, \tag{2.27}$$

where $\varepsilon = \operatorname{sgn} s_m$ and $\tau(\lambda)$ satisfies one of the following conditions:

- (i) *if $\ell = 2m$ then $\tau(\lambda) = o(\lambda)$, $\lambda \widehat{\rightarrow} \infty$, and in* (2.24) *$s_{2m+1}$ can be an arbitrary real number;*
- (ii) *if $\ell = 2m+1$ then $\tau(\lambda) = o(1)$ as $\lambda \widehat{\rightarrow} \infty$;*
- (iii) *if $\ell > 2m+1$ then $\tau(\lambda)$ has the asymptotic expansion*

$$\tau(\lambda) = -\frac{\widehat{s}_0}{\lambda} - \cdots - \frac{\widehat{s}_{\ell-2m-2}}{\lambda^{\ell-2m-1}} + o\left(\frac{1}{\lambda^{\ell-2m-1}}\right), \quad \lambda \widehat{\rightarrow} \infty, \tag{2.28}$$

where the sequence $(\widehat{\mathbf{s}}, \ell - 2m - 2)$ is determined by the matrix equation

$$T(p_{m+1}, \dots, p_0, -\varepsilon\widehat{s}_0, \dots, -\varepsilon\widehat{s}_{\ell-2m-2})\, T(s_m, \dots, s_\ell) = s_m I_{\ell-m+1}.$$

Proof. It is clear that the function φ admits the asymptotic expansion (2.26) if and only if $c(\lambda) := -\lambda^{m+1}\varphi(\lambda)$ admits the asymptotic expansion of the form (2.21) with $n = \ell - m$ and, moreover, by standard inversion of expansions, this is equivalent for

$d(\lambda) := 1/c(\lambda)$ to admit the asymptotic expansion of the form (2.21) with $n = \ell - m$. Now by substituting the expansions for the terms in the formula

$$\left(\frac{p(\lambda)+\varepsilon\tau(\lambda)}{\lambda^{m+1}}\right)\left(-\lambda^{m+1}\varphi(\lambda)\right) = s_m \tag{2.29}$$

and applying Lemma 2.11 it is seen that

$$\frac{p(\lambda)+\varepsilon\tau(\lambda)}{\lambda^{m+1}} = p_{m+1} + \frac{p_m}{\lambda} + \cdots + \frac{p_{2m-\ell+1}}{\lambda^{\ell-m}} + o\left(\frac{1}{\lambda^{\ell-m}}\right), \quad \lambda\widehat{\to}\infty, \tag{2.30}$$

where the coefficients p_j are determined by the matrix equation

$$T(p_{m+1},\dots,p_{2m-\ell+1})\,T(s_m,\dots,s_\ell) = s_m I_{\ell-m+1}. \tag{2.31}$$

In particular, $p(\lambda)$ appearing in (2.27) is a polynomial of degree $m+1$ whose coefficients satisfy the matrix equality (2.25) when $\ell \geq 2m$. Hence, $p(\lambda)$ in (2.27) can be taken to be the polynomial defined by (2.24) in Lemma 2.12.

(i) If $\ell = 2m$ then the formula (2.30) shows that the function $\tau(\lambda)$ in (2.27) satisfies $\tau(\lambda) = o(\lambda)$ as $\lambda\widehat{\to}\infty$, independent from the selection of the real number s_{2m+1} in (2.24).

(ii) If $\ell = 2m+1$ then the formula (2.30) shows that $\tau(\lambda) = o(1)$ as $\lambda\widehat{\to}\infty$.

(iii) If $\ell > 2m+1$ then the formula (2.30) shows that the function $\tau(\lambda)$ in (2.27) has the asymptotic expansion (2.28) with $\widehat{s}_j = -\varepsilon p_{-j-1}$, $j = 0,\dots,\ell-2m-2$, with coefficients p_j as in (2.31). $\square$

Associate with the expansion (2.26) the $(m+1)\times(m+1)$ matrix S_m as in (1.4). Then

$$S_m = \begin{pmatrix} & & s_m \\ & \iddots & \vdots \\ s_m & \dots & s_{2m} \end{pmatrix} = J_{m+1} T(s_m,\dots,s_{2m}), \tag{2.32}$$

where $T(\mathbf{s}, m, 2m)$ and J_{m+1} are as in (2.19) and (2.20). Let the monic polynomial $p(\lambda)$ of degree $m+1$ be defined by the formula (2.24), where s_{2m+1} is an arbitrary real number in the case $\ell = 2m$. Then it follows from (2.25) that

$$T(p_{m+1},\dots,p_1)J_{m+1} = s_m S_m^{-1}$$

and hence

$$s_m \frac{p(\lambda)-p(\bar\omega)}{\lambda-\bar\omega} = \begin{pmatrix} 1 & \dots & \lambda^m \end{pmatrix} s_m^2 S_m^{-1} \begin{pmatrix} 1 \\ \vdots \\ \bar\omega^m \end{pmatrix}, \tag{2.33}$$

so that $s_m p \in \mathbf{N}_{\nu_-(S_m)}$ (as $s_m \neq 0$), i.e., the negative index of the generalized Nevanlinna function $s_m p(\lambda)$ is equal to $\nu_-(S_m)$, the number of negative eigenvalues of the matrix S_m.

When the function $\varphi(\lambda)$ in (2.26) is a generalized Nevanlinna function the statements in the previous lemma can be specified further. The next result shows how the classes $\mathbf{N}_{\kappa,-\ell}$ behave under linear fraction transforms; for this it suffices to consider the transform $\varphi(\lambda) \to -1/\varphi(\lambda)$; as in Lemma 2.13 the result is expressed via the function $\tau(\lambda)$ which will appear in later sections, too.

Lemma 2.14. *Let the notations and assumptions be as in Lemma 2.13. Then the following assertions hold:*

(i) *$\varphi \in \mathbf{N}_\kappa$ if and only if $\tau \in \mathbf{N}_{\kappa-\nu_-(S_m)}$;*
(ii) *$\varphi \in \mathbf{N}_{\kappa,-2m}$ if and only if $\tau \in \mathbf{N}_{\kappa-\nu_-(S_m)}$ and $\tau(\lambda) = o(\lambda)$ as $\lambda\widehat{\to}\infty$;*
(iii) *$\varphi \in \mathbf{N}_{\kappa,-2m-1}$ if and only if $\tau \in \mathbf{N}_{\kappa-\nu_-(S_m),1}$ and $\tau(\lambda) = o(1)$ as $\lambda\widehat{\to}\infty$;*
(iv) *$\varphi \in \mathbf{N}_{\kappa,-\ell}$ with $\ell > 2m+1$ if and only if $\tau \in \mathbf{N}_{\kappa-\nu_-(S_m),-(\ell-2m-2)}$ and $\tau(\lambda) = o(1)$ as $\lambda\widehat{\to}\infty$.*

Proof. (i) The condition $\varphi \in \mathbf{N}_\kappa$ is equivalent to $-1/\varphi \in \mathbf{N}_\kappa$ ($\varphi \neq 0$). Since $\ell \geq 2m$ it follows from Lemma 2.13 that $\tau(\lambda) = o(\lambda)$, $\lambda\widehat{\to}\infty$. Hence the definition of $\tau(\lambda)$ in (2.27) and [13, Theorem 4.1, Corollary 4.2] imply that $-1/\varphi \in \mathbf{N}_\kappa$ if and only if $\tau(\lambda)$ is a generalized Nevanlinna function such that its negative index $\kappa(\tau)$ satisfies

$$\kappa = \kappa(-1/\varphi) = \kappa(\varepsilon P) + \kappa(\tau) = \nu_-(S_m) + \kappa(\tau);$$

see (2.33). Hence, $\varphi \in \mathbf{N}_\kappa$ if and only if $\tau \in \mathbf{N}_{\kappa-\nu_-(S_m)}$.

(ii) The statement for $\ell = 2m$ is obtained now directly from part (i) of Lemma 2.13.

(iii) & (iv) Let $\ell = 2k$ or $\ell = 2k-1$ with $k > m$ and rewrite the expansion of $\varphi(\lambda)$ in (2.26) as follows:

$$\varphi(\lambda) = -\frac{s_m}{\lambda^{m+1}} - \cdots - \frac{s_{2k-1}}{\lambda^{2k}} + \frac{C(\lambda)}{\lambda^{2k}}, \quad C(\lambda) = o\,(1)\,, \qquad \lambda\widehat{\to}\infty. \tag{2.34}$$

Then by (2.13) $\varphi \in \mathbf{N}_{\kappa,-2k+1}$ if and only if $C \in \mathbf{N}_{\kappa',1}$ and, similarly, $\varphi \in \mathbf{N}_{\kappa,-2k}$ if and only if $C \in \mathbf{N}_{\kappa',0}$ for some $\kappa' \leq \kappa$. Now the expansion in (2.30) can be rewritten as follows

$$\frac{p(\lambda) + \varepsilon\tau(\lambda)}{\lambda^{m+1}} = p_{m+1} + \frac{p_m}{\lambda} + \cdots + \frac{p_{2(m-k+1)}}{\lambda^{2k-m-1}} + \frac{D(\lambda)}{\lambda^{2k-m-1}}, \quad \lambda\widehat{\to}\infty, \tag{2.35}$$

where $D(\lambda)$ satisfies

$$D(\lambda) = \frac{p_{m+1}}{s_m}\, C(\lambda) + O\left(\frac{1}{\lambda}\right), \quad \lambda\widehat{\to}\infty;$$

compare [21, Lemma 4.1]. The formula (2.35) is equivalent to the following expansion for $\tau(\lambda)$:

$$\tau(\lambda) = \varepsilon\left(\frac{p_{-1}}{\lambda} + \cdots + \frac{p_{2(m-k+1)}}{\lambda^{2(k-m-1)}}\right) + \frac{\varepsilon D(\lambda)}{\lambda^{2(k-m-1)}}, \tag{2.36}$$

where $\varepsilon D(\lambda) = \frac{\varepsilon p_{m+1}}{s_m}\, C(\lambda) + O\left(\frac{1}{\lambda}\right)$ as $\lambda\widehat{\to}\infty$ (and for $k = m+1$ the first term in the righthand side of (2.36) is missing). Since here $\varepsilon D(\lambda) \in \mathbf{N}_{\kappa'',j}$ for some $\kappa'' \in \mathbb{N}$ is equivalent to $C \in \mathbf{N}_{\kappa',j}$ for $j = 0,1$ ($p_{m+1} = 1$), the assertion $\tau \in \mathbf{N}_{\kappa-\nu_-(S_m),-(\ell-2m-2)}$ for the values $\ell = 2k$ and $\ell = 2k-1$ with $k > m$ follows from (2.36) by the inclusions (2.13). $\square$

In the special case $\kappa = 0$ and $m = 0$ the result in Proposition 2.14 implies [21, Theorem 4.2]. Results analogous to that in Proposition 2.14 in the special case where $\ell = 2k$ is even can be found from [13, Proposition 5.4], [14, Theorem 5.4]; the even case ($\ell = 2k$) is easier, since then the expansion (2.26) for $\varphi \in \mathbf{N}_\kappa$ is equivalent to $\varphi \in \mathbf{N}_{\kappa,-2m}$, see e.g. [14, Corollaries 3.4, 3.5].

3. Basic moment and interpolation problems

In this section solutions of the so-called basic moment and interpolation problems will be described. The treatment is divided into two cases: nondegenerate and degenerate problems according to $\det S_n \neq 0$ and $\det S_n = 0$, where the matrix S_n is as defined in (1.4). The even and the odd cases of the problems $MP_\kappa(\mathbf{s}, \ell)$

and $IP_\kappa(\mathbf{s},\ell)$ will be treated in a parallel way. In the even case ($\ell = 2n$) all the statements will be formulated only for the problem $MP_\kappa(\mathbf{s},2n)$, since the problems $MP_\kappa(\mathbf{s},2n)$ and $IP_\kappa(\mathbf{s},2n)$ are equivalent (see Remark 2.9). In what follows a sequence $(\mathbf{s},\ell) := \{s_i\}_{i=0}^{\ell}$ is said to be *normalized* if the first nonzero element of $\mathbf{s}$ has modulus 1.

3.1. **Nondegenerate basic moment and interpolation problems.** Let $n = [\ell/2]$, so that either $\ell = 2n$ or $\ell = 2n+1$. Nondegenerate problems $MP_\kappa(\mathbf{s},\ell)$ and $IP_\kappa(\mathbf{s},\ell)$ are said to be *basic* if the sequence $(\mathbf{s},\ell)$ is normalized and $\det S_j = 0$ for all $j \le n-1$, or, equivalently, if

$$s_0 = s_1 = \cdots = s_{n-1} = 0, \quad |s_n| = 1. \tag{3.1}$$

Thus a function $\varphi \in \mathbf{N}_\kappa$ is a solution to the nondegenerate basic interpolation problem $IP_\kappa(\mathbf{s},2n)$, if it satisfies the condition

$$\varphi(\lambda) = -\frac{s_n}{\lambda^{n+1}} - \frac{s_{n+1}}{\lambda^{n+2}} - \cdots - \frac{s_\ell}{\lambda^{\ell+1}} + o\left(\frac{1}{\lambda^{\ell+1}}\right), \quad \lambda \widehat{\to} \infty. \tag{3.2}$$

If $IP_\kappa(\mathbf{s},\ell)$ is nondegenerate and basic, then the Hankel matrix S_n has the form

$$S_n = \begin{pmatrix} & & s_n \\ & \cdot^{\cdot^{\cdot}} & \vdots \\ s_n & \ldots & s_{2n} \end{pmatrix},$$

where all the nonspecified entries are equal to 0. Define the monic polynomial $p(\lambda)$ of degree $n+1$ by the formula (2.24), where $m = n$ and s_{2n+1} is an arbitrary real number in the case of even ℓ. Then it follows from (2.33) that $s_n p \in \mathbf{N}_{\nu_-(S_n)}$. In the case of even $\ell = 2n$ the following result is a corollary of general descriptions of $\mathcal{M}_\kappa(\mathbf{s},\ell)$ given in [18] and [11], and a short proof in this even case $\ell = 2n$ has been presented in [9]. Here the result both for even and odd ℓ is an immediate consequence of the general transformation result given in Proposition 2.14.

Lemma 3.1. *Let $(\mathbf{s},\ell)$ be a sequence of real numbers such that (3.1) holds with $n = [\ell/2]$, let $\nu_- := \nu_-(S_n)$, let $p \in \mathbf{N}_{\nu_-}$ be the polynomial of degree $n+1$ defined in (2.24), and $\varepsilon = s_n$. Then $MP_\kappa(\mathbf{s},\ell)$ and $IP_\kappa(\mathbf{s},\ell)$ are solvable if and only if*

$$\kappa \ge \nu_-. \tag{3.3}$$

If $\kappa \ge \nu_-$ then the formula

$$\varphi(\lambda) = -\frac{\varepsilon}{p(\lambda) + \varepsilon\tau(\lambda)}, \tag{3.4}$$

describes the sets $\mathcal{M}_\kappa(\mathbf{s},\ell)$ and $\mathcal{I}_\kappa(\mathbf{s},\ell)$ as follows: in the even case

$$\varphi \in \mathcal{M}_\kappa(\mathbf{s},\ell) \Leftrightarrow \tau \in \mathbf{N}_{\kappa-\nu_-} \text{ and satisfies (E);}$$

and in the odd case

$$\varphi \in \mathcal{M}_\kappa(\mathbf{s},\ell) \Leftrightarrow \tau \in \mathbf{N}_{\kappa-\nu_-,1} \text{ and satisfies (O);}$$

$$\varphi \in \mathcal{I}_\kappa(\mathbf{s},\ell) \Leftrightarrow \tau \in \mathbf{N}_{\kappa-\nu_-} \text{ and satisfies (O).}$$

Proof. Let $\varphi \in \mathcal{I}_\kappa(\mathbf{s},\ell)$. Then $\varphi \in \mathbf{N}_\kappa$ with the expansion (3.2). Since $|s_n| = 1$ Proposition 2.14 shows that now equivalently

$$-1/\varphi(\lambda) = \varepsilon p(\lambda) + \tau(\lambda), \tag{3.5}$$

where $\tau \in \mathbf{N}_{\kappa-\nu_-}$. In particular, the solvability criterion (3.3) and the assertions in the even case $\ell = 2n$ and the odd case $\ell = 2n+1$ are obtained from and parts (i) and (ii) of Lemma 2.13, respectively, and Lemma 2.14 (i).

The last statement for the moment problem $MP_\kappa(\mathbf{s}, \ell)$ in the odd case is implied by Lemma 2.14 (iii). □

3.2. **Degenerate basic problems.** Let $\ell \in \mathbb{N}$ and $n = [\ell/2]$. Degenerate moment and interpolation problems $MP_\kappa(\mathbf{s}, \ell)$ and $IP_\kappa(\mathbf{s}, \ell)$ are said to be *basic* if $\det S_j = 0$ for all $j \le n$ and the sequence $(\mathbf{s}, \ell)$ is normalized. Consequently, the set of degenerate basic moment problems can be divided into two cases as follows:

(A) $s_j = 0$ for all $j = 0, \dots, 2n$.
(B) There is at least one nonzero moment s_j for some $j = n+1, \dots, 2n$. Let m be the minimal number for which $|s_m| = 1$ $(n < m \le 2n)$.

3.2.1. *Degenerate basic problems: Case (A).* In this case $s_0 = \cdots = s_{2n} = 0$ and $\nu_0(S_n) = n+1$. Let us denote $\nu_0 := \nu_0(S_n)$. In the next theorem descriptions of the sets of solutions to the degenerate basic problems $MP_\kappa(\mathbf{s}, \ell)$ and $IP_\kappa(\mathbf{s}, \ell)$ are given.

Lemma 3.2. *Let $(\mathbf{s}, \ell)$ be a sequence of real numbers satisfying the assumption (A) with $n = [\ell/2]$. Then in the even case the problems $MP_\kappa(\mathbf{s}, \ell)$ and $IP_\kappa(\mathbf{s}, \ell)$ are solvable if and only if*

$$\text{either } \kappa = 0, \text{ or } \kappa \ge \nu_0 := \nu_0(S_n). \tag{3.6}$$

In the odd case $MP_\kappa(\mathbf{s}, \ell)$ and $IP_\kappa(\mathbf{s}, \ell)$ are solvable if and only if

$$\text{either } \kappa = 0 \text{ and } s_\ell = 0, \text{ or } \kappa \ge \nu_0. \tag{3.7}$$

If $\kappa = 0$ (and $s_\ell = 0$ in the odd case), then it has the unique solution $\varphi(\lambda) \equiv 0$.

If $\kappa \ge \nu_0$ and ν is given by (2.17)*, then the formula*

$$\varphi(\lambda) = \frac{\widehat{\varphi}(\lambda)}{\lambda^{2\nu_0}}, \tag{3.8}$$

describes the sets $\mathcal{M}_\kappa(\mathbf{s}, \ell)$ and $\mathcal{I}_\kappa(\mathbf{s}, \ell)$ as follows: in the even case

$$\varphi \in \mathcal{M}_\kappa(\mathbf{s}, \ell) \Leftrightarrow \widehat{\varphi} \in \mathbf{N}_{\kappa-\nu} \text{ and satisfies (E);}$$

and in the odd case

$$\varphi \in \mathcal{M}_\kappa(\mathbf{s}, \ell) \Leftrightarrow \widehat{\varphi} + s_\ell \in \mathbf{N}_{\kappa-\nu_-,1} \text{ and satisfies (O)};$$
$$\varphi \in \mathcal{I}_\kappa(\mathbf{s}, \ell) \Leftrightarrow \widehat{\varphi} + s_\ell \in \mathbf{N}_{\kappa-\nu_-} \text{ and satisfies (O)}.$$

Proof. The case $\kappa = 0$ is trivial, since if $\varphi \in \mathbf{N}_0$, then $s_0 = 0$ implies $\varphi = 0$.

Let $\kappa > 0$ and let $\varphi \in \mathbf{N}_\kappa$ be a solution of the interpolation problem $IP_\kappa(\mathbf{s}, \ell)$. Then it follows from (3.2) and (A) that in the even case $(\ell = 2n)$

$$\varphi(\lambda) = o\left(\frac{1}{\lambda^{2n+1}}\right), \quad \lambda \widehat{\to} \infty. \tag{3.9}$$

and in the odd case

$$\varphi(\lambda) = -\frac{s_{2n+1}}{\lambda^{2n+2}} + o\left(\frac{1}{\lambda^{2n+2}}\right), \quad \lambda \widehat{\to} \infty. \tag{3.10}$$

In both cases

$$\lim_{\lambda \widehat{\to} \infty} \lambda^{2n+1} \varphi(\lambda) = 0,$$

and, hence, the multiplicity $\pi_\infty(\varphi)$ is at least $n+1$. In view of Remark 2.1 one has

$$\kappa \geq \pi_\infty(\varphi) \geq n+1 = \nu_0.$$

This proves the necessity of the condition $\kappa \geq \nu_0$. Due to Lemma 2.10 φ admits the representation (3.8) with $\widehat{\varphi} \in \mathbf{N}_{\kappa-\nu}$. In addition, it follows from (3.9) and (3.10) that $\widehat{\varphi}$ satisfy the assumption (E) if ℓ is even, and $\widehat{\varphi} + s_\ell$ satisfy (O) if ℓ is odd.

And finally, let $\ell = 2n+1$ and $\kappa \geq \nu_0 = n+1$. Then by Lemma 2.10 $\varphi \in \mathbf{N}_{\kappa,-\ell}$ if and only if $\widehat{\varphi} \in \mathbf{N}_{\kappa-\nu,1}$. This completes the proof. □

3.2.2. *Degenerate basic problems: Case (B).* In this case a function φ from $\mathbf{N}_{\kappa,\ell}$ ($\mathbf{N}_{\kappa,-\ell}$) is a solution to the degenerate basic problem $MP_\kappa(\mathbf{s},\ell)$ ($IP_\kappa(\mathbf{s},\ell)$), if

$$\varphi(\lambda) = -\frac{s_m}{\lambda^{m+1}} - \cdots - \frac{s_\ell}{\lambda^{\ell+1}} + o\left(\frac{1}{\lambda^{\ell+1}}\right), \quad \lambda \widehat{\to} \infty, \tag{3.11}$$

with $|s_m| = 1$ and $n < m \leq 2n$, where $n = [\ell/2]$.

Lemma 3.3. *Let $(\mathbf{s},\ell)$ be a sequence of real numbers satisfying the assumption (B) with $n = [\ell/2]$. Then the problems $MP_\kappa(\mathbf{s},\ell)$ and $IP_\kappa(\mathbf{s},\ell)$ are solvable if and only if*

$$\kappa \geq k := \nu_0 + \nu_-, \quad \nu_0 := \nu_0(S_n),\ \nu_- := \nu_-(S_n). \tag{3.12}$$

Let the sequence $(\widehat{\mathbf{s}}, \ell - 2\nu_0) = \{\widehat{s}_i\}_{i=0}^{\ell-2\nu_0}$ be given by the equalities

$$\widehat{s}_j = s_{j+2\nu_0}, \quad (j = 0, \dots, \ell - 2\nu_0), \tag{3.13}$$

and let ν be defined by (2.17). Then the formula (3.8) establishes a one-to-one correspondence between solutions φ of the problem $IP_\kappa(\mathbf{s},\ell)$ and solutions $\widehat{\varphi}$ of the nondegenerate basic problem $IP_{\kappa-\nu}(\widehat{s}, \ell-2\nu_0)$. A similar statement concerning the problems $MP_\kappa(\mathbf{s},\ell)$ and $MP_{\kappa-\nu}(\widehat{s}, \ell - 2\nu_0)$) is also true.

Proof. Let $\varphi \in \mathcal{I}_\kappa(\mathbf{s},\ell)$, so that φ belongs to $\mathbf{N}_\kappa$ and satisfies (3.11). Now the matrix S_n takes the form

$$S_n = \begin{pmatrix} \mathbf{0}_{(m-n)\times(m-n)} & \mathbf{0}_{(m-n)\times(2n-m+1)} \\ \mathbf{0}_{(2n-m+1)\times(m-n)} & S_{[m-n,n]} \end{pmatrix},$$

where

$$S_{[m-n,n]} = (s_{i+j})_{i,j=m-n}^n = \begin{pmatrix} & & s_m \\ & \iddots & \vdots \\ s_m & \dots & s_{2n} \end{pmatrix}$$

is invertible since $s_m \neq 0$ ($|s_m| = 1$). It is clear that

$$\nu_0(S_n) = m - n > 0.$$

To determine the index $\nu_-(S_n)$ consider the following three subcases:

(B1) m is even and $s_m > 0$ (denote $m = 2k$);
(B2) m is odd (denote $m = 2k-1$);
(B3) m is even and $s_m < 0$ (denote $m = 2k-2$).

Then one can easily check that

$$\nu_-(S_n) = \begin{cases} n-k, & \text{in case (B1);} \\ n-k+1, & \text{in case (B2);} \\ n-k+2, & \text{in case (B3),} \end{cases}$$

so that in each of the cases (B1)–(B3) one has

$$\nu_0(S_n)+\nu_-(S_n)=k>0. \tag{3.14}$$

It follows from (3.11) that in the case (B1)

$$\lim_{\lambda\hat{\to}\infty}\lambda^{2k+1}\varphi(\lambda)=-s_{2k}<0,\quad \lim_{\lambda\hat{\to}\infty}\lambda^{2k-1}\varphi(\lambda)=0,$$

In the cases (B2) and (B3), respectively, one obtains

$$\lim_{\lambda\hat{\to}\infty}\lambda^{2k+1}\varphi(\lambda)=\infty,\quad \lim_{\lambda\hat{\to}\infty}\lambda^{2k-1}\varphi(\lambda)=0;$$
$$\lim_{\lambda\hat{\to}\infty}\lambda^{2k+1}\varphi(\lambda)=\infty,\quad \lim_{\lambda\hat{\to}\infty}\lambda^{2k-1}\varphi(\lambda)=-s_{2k-2}>0.$$

Hence, in each of these cases, ∞ is a GZNT of $\varphi(\lambda)$ of multiplicity $\pi_\infty(\varphi)=k$; see (2.4). By Theorem 2.2 (or Remark 2.1) this implies the inequality (3.12).

Due to Lemma 2.10 φ admits the representation (3.8) where $\widehat{\varphi}\in\mathbf{N}_{\kappa-\nu}$. Clearly, the expansion (3.11) can be rewritten as

$$\widehat{\varphi}(\lambda)=-\frac{s_m}{\lambda^{m-2\nu_0+1}}-\cdots-\frac{s_\ell}{\lambda^{\ell-2\nu_0+1}}+o\left(\frac{1}{\lambda^{\ell-2\nu_0+1}}\right),\quad \lambda\widehat{\to}\infty,$$

Therefore, $\widehat{\varphi}\in\mathcal{I}_{\kappa-\nu}(\widehat{s},\ell-2\nu_0)$. These arguments can be reversed to obtain the converse statement.

In the case where ℓ is odd it follows from Lemma 2.10 that $\varphi\in\mathbf{N}_{\kappa,-\ell}$ if and only if $\widehat{\varphi}\in\mathbf{N}_{\kappa-\nu,-(\ell-2\nu_0)}$. This proves the statement concerning the set $\mathcal{M}_\kappa(\mathbf{s},\ell)$. $\square$

The following Lemma summarizes the results of Lemmas 3.1 and 3.3.

Lemma 3.4. *Let $(\mathbf{s},\ell)$ be a sequence of real numbers satisfying the assumption (B) with $n=[\ell/2]$, let (3.12) hold, let ν be defined by (2.17), $\varepsilon=s_m$, and let*

$$\widehat{p}(\lambda)=\frac{1}{\det S_{[m-n,n]}}\det\begin{pmatrix} & & s_m & s_{m+1}\\ & \iddots & \iddots & \vdots\\ s_m & s_{m+1} & \dots & s_{2n+1}\\ 1 & \lambda & \dots & \lambda^{2n+1}\end{pmatrix}, \tag{3.15}$$

where s_{2n+1} is an arbitrary real number if ℓ is even. Then the formula

$$\varphi(\lambda)=-\frac{\varepsilon}{\lambda^{2\nu_0}(\widehat{p}(\lambda)+\varepsilon\tau(\lambda))}, \tag{3.16}$$

describes the sets $\mathcal{M}_\kappa(\mathbf{s},\ell)$ and $\mathcal{I}_\kappa(\mathbf{s},\ell)$ as follows: in the even case

$$\varphi\in\mathcal{M}_\kappa(\mathbf{s},\ell)\Leftrightarrow\widehat{\varphi}\in\mathbf{N}_{\kappa-\nu_--\nu}\text{ and satisfies (E);}$$

and in the odd case

$$\varphi\in\mathcal{M}_\kappa(\mathbf{s},\ell)\Leftrightarrow\widehat{\varphi}+s_\ell\in\mathbf{N}_{\kappa-\nu_--\nu,1}\text{ and satisfies (O)};$$
$$\varphi\in\mathcal{I}_\kappa(\mathbf{s},\ell)\Leftrightarrow\widehat{\varphi}+s_\ell\in\mathbf{N}_{\kappa-\nu_--\nu}\text{ and satisfies (O)}.$$

Proof. Assume that (3.12) holds. In Lemma 3.3 the problem $IP_\kappa(\mathbf{s},\ell)$ was reduced to the problem $IP_\kappa(\widehat{s},\ell-2\nu_0)$. By Lemma 3.1 the set $\mathcal{I}_{\kappa-\nu}(\widehat{s},\ell-2\nu_0)$ can be described by the formula

$$\widehat{\varphi}(\lambda)=-\frac{\varepsilon}{\widehat{p}(\lambda)+\varepsilon\tau(\lambda)} \tag{3.17}$$

where τ is a function from the class $\mathbf{N}_{\kappa-(\nu+\nu_-)}$ such that the appropriate condition (E) or (O) is satisfied. Substitution of (3.17) into (3.8) yields (3.16).

Due to Lemma 3.3 and Lemma 3.1 φ belongs to $\mathcal{M}_\kappa(\mathbf{s},\ell)$ with odd ℓ, if and only if the function τ belongs to $\mathbf{N}_{\kappa-(\nu+\nu_-)}$ and satisfies the condition (O). □

4. Schur algorithm

The present approach to the degenerate moment problem is based on the following reduction algorithm which for the nondegenerate case with even index ℓ was considered in [9].

4.1. **One step reduction for moment problems which are not basic.** Let $\mathbf{s} = \{s_j\}_{j=0}^{\ell}$ be an arbitrary normalized sequence of real numbers and let $S_n = (s_{i+j})_{i,j=0}^{n}$ be the Hankel matrix as defined in (1.4). Assume that $S_n \neq 0$ and consider a sequence of *normal indices* of S_n,

$$0 < n_1 < \cdots < n_N \le n+1 \tag{4.1}$$

which are characterized by the conditions

$$\det S_{n_j-1} \neq 0 \quad (j = 1, \dots, N). \tag{4.2}$$

In particular, the first normal index n_1 is the minimal natural number such that $\det S_{n_1-1} \neq 0$, or , equivalently, that

$$s_0 = s_1 = \cdots = s_{n_1-2} = 0, \quad s_{n_1-1} \neq 0. \tag{4.3}$$

Note that the first normal index satisfies $n_1(= n_N) = n+1$ precisely when the moment problem is nondegenerate and basic and that there are no normal indices for moment problems which are degenerate and basic; see Section 3.

In this section it is assumed that the moment problem is not basic, i.e., one has $n_1 \le n$. Let the sequence $(\mathbf{s},\ell) = \{s_j\}_{j=0}^{\ell}$ be normalized and denote

$$\varepsilon_1 = \operatorname{sgn} s_{n_1-1} = \pm 1.$$

In this case a function $\varphi \in \mathbf{N}_{\kappa,-\ell}$ is a solution to the moment problem $\mathcal{M}_\kappa(\mathbf{s},\ell)$ if

$$\varphi(\lambda) = -\frac{s_{n_1-1}}{\lambda^{n_1}} - \frac{s_{n_1}}{\lambda^{n_1+1}} - \cdots - \frac{s_\ell}{\lambda^{\ell+1}} + o\left(\frac{1}{\lambda^{\ell+1}}\right), \quad \lambda \widehat{\to} \infty. \tag{4.4}$$

Then $-1/\varphi \in \mathbf{N}_\kappa$ and, moreover, by part (iii) of Lemma 2.13 $-1/\varphi$ admits the representation

$$-1/\varphi(\lambda) = \varepsilon_1 p_1(\lambda) + a_1^2 \varphi_1(\lambda), \tag{4.5}$$

where $p_1(\lambda) = p_{n_1}^{(1)}\lambda^{n_1} + \cdots + p_0^{(1)}$ is a monic polynomial of degree n_1 ($p_{n_1}^{(1)} = 1$), defined by the equation (2.24) with $m = n_1 - 1$, and $a_1 (> 0)$ is chosen in such a way that the sequence $(\mathbf{s}^{(1)}, \ell - 2n_1) = (s_i^{(1)})_{i=0}^{\ell-2n_1}$ defined by the expansion of $\varphi_1(\lambda)$

$$\varphi_1(\lambda) = -\frac{s_0^{(1)}}{\lambda} - \frac{s_1^{(1)}}{\lambda^2} - \cdots - \frac{s_{\ell-2n_1}^{(1)}}{\lambda^{\ell-2n_1+1}} + o\left(\frac{1}{\lambda^{\ell-2n_1+1}}\right) \quad (\lambda \widehat{\to} \infty), \tag{4.6}$$

is normalized. Moreover, by Proposition 2.14 (iii) φ_1 is a generalized Nevanlinna function from the class

$$\mathbf{N}_{\kappa-\kappa_1, -(\ell-2n_1)}, \quad \kappa_1 := \nu_-(S_{n_1-1}).$$

As was shown in Lemma 2.13 the moment sequence $(\mathbf{s}^{(1)}, \ell - 2n_1)$ is uniquely defined by the matrix equations

$$T(s_{n_1-1}, \dots, s_{j+2n_1}) T(p_{n_1}^{(1)}, \dots, p_0^{(1)}, -\varepsilon_1 a_1^2 s_0^{(1)}, \dots, -\varepsilon_1 a_1^2 s_j^{(1)}) = \varepsilon_1 I_{j+n_1+2}, \tag{4.7}$$

where $0 \le j \le \ell - 2n_1$.

The above considerations yield the following result.

Proposition 4.1. *Let S_n be a Hankel matrix, let n_1 be the first normal index of S_n, $n_1 \le n$, let the monic polynomial $p_1(\lambda) = p_{n_1}^{(1)}\lambda^{n_1} + \dots + p_0^{(1)}$ and the induced moment sequence $(\mathbf{s}^{(1)}, \ell - 2n_1)$ be defined by (4.7), $\varepsilon_1 = s_{n_1-1}$, $\kappa_1 := \nu_-(S_{n_1-1})$. Then the formula*

$$\varphi(\lambda) = T_1[\varphi_1(\lambda)] := \frac{-\varepsilon_1}{p_1(\lambda) + \varepsilon_1 a_1^2 \varphi_1(\lambda)} \tag{4.8}$$

establishes a one-to-one correspondence between the sets $\mathcal{M}_\kappa(\mathbf{s}, \ell)$ and $\mathcal{M}_{\kappa-\kappa_1}(\mathbf{s}^{(1)}, \ell - 2n_1)$ as well as between the sets $\mathcal{I}_\kappa(\mathbf{s}, \ell)$ and $\mathcal{I}_{\kappa-\kappa_1}(\mathbf{s}^{(1)}, \ell - 2n_1)$.

The normal indices of the induced Hankel matrix $S_{n-n_1}^{(1)} = (s_{i+j}^{(1)})_{i,j=0}^{n-n_1}$ can be derived from the normal indices of the original Hankel matrix S_n. This is given in the next Proposition.

Proposition 4.2. *Let $n_1 < n_2 < \dots < n_N (\le N+1)$ be all normal indices of the Hankel matrix S_n. Then the normal indices of the induced Hankel matrix $S_{n-n_1}^{(1)}$ are*

$$n_2 - n_1 < \dots < n_N - n_1.$$

Proof. It follows from (A.3) in Lemma A.3 that $\det S_{i-n_1}^{(1)} \ne 0$ if and only if $i = n_2, \dots, n_N$ $(n_1 \le i \le n)$. □

Now applying Proposition 4.1 to the matrix $S_{n-n_1}^{(1)}$ one construct a polynomial p_2 and a Hankel matrix $S_{n-n_2}^{(2)}$. After N inductive steps one obtains the Hankel matrix $S_{n-n_N}^{(N)}$ of induced moments and subsequent application of Lemma A.3 yields the following

Corollary 4.3. *Let $S_{n-n_j}^{(j)}$ be the Hankel matrix of induced moments after j steps $(1 \le j \le N)$. Then the set of normal indices of the Hankel matrix $S_{n-n_j}^{(j)}$ takes the form $\{n_k - n_j\}_{k=j+1}^N$. Moreover, for all i such that $n_j \le i \le n$ one has*

$$\nu_\pm(S_{i-n_j}^{(j)}) = \nu_\pm(S_i) - \nu_\pm(S_{n_j-1}); \tag{4.9}$$

$$\nu_0(S_{i-n_j}^{(j)}) = \nu_0(S_i). \tag{4.10}$$

In particular, the matrix $S_{n-n_N}^{(N)}$ has no normal indices anymore, that is $\det S_{i-n_N}^{(N)} = 0$ for all i such that $n_N \le i \le n$.

Proof. The first statement is implied by the formula (A.3) in Lemma A.3. The formula (4.9) can be obtained by induction. Indeed, for $j = 1$ the statement is contained in (A.2) of Lemma A.3. Assume that (4.9) holds for some j $(1 \le j \le N)$ and all i such that $n_j \le i \le n$. Then it follows from (A.2) that

$$\nu_\pm(S_{i-n_{j+1}}^{(j+1)}) = \nu_\pm(S_{i-n_j}^{(j)}) - \nu_\pm(S_{n_{j+1}-n_j-1}^{(j)}).$$

In view of the induction assumption this yields

$$\begin{aligned}\nu_\pm(S_{i-n_{j+1}}^{(j+1)}) &= \nu_\pm(S_i) - \nu_\pm(S_{n_j-1}) - (\nu_\pm(S_{n_{j+1}-1}) - \nu_\pm(S_{n_j-1})) \\ &= \nu_\pm(S_i) - \nu_\pm(S_{n_{j+1}-1}).\end{aligned}$$

The formula (4.10) is immediate from (A.3) in Lemma A.3. □

4.2. **Algorithm.** Let us define a sequence $\kappa_1 \le \cdots \le \kappa_N$ by the equalities

$$\kappa_j = \nu_-(S_{n_j-1}), \quad j = 1, \dots, N. \tag{4.11}$$

Due to Proposition 4.1 on each step one obtains a linear fractional transformation

$$\varphi_{j-1}(\lambda) = \mathcal{T}_j[\varphi_j(\lambda)] := \frac{-\varepsilon_j}{p_j(\lambda) + \varepsilon_j a_j^2 \varphi_j(\lambda)}, \tag{4.12}$$

where

$$\varepsilon_j = \operatorname{sign} s^{(j-1)}_{n_j - n_{j-1} - 1} (= \pm 1), \quad a_j > 0 \quad (0 \le j \le N-1).$$

The transformation $\mathcal{T}_j$ establishes a one-to-one correspondence between the sets $\varphi_{j-1} \in \mathcal{M}_{\kappa-\kappa_{j-1}}(\mathbf{s}^{(j-1)}, \ell - 2n_{j-1})$ and $\varphi_j \in \mathcal{M}_{\kappa-\kappa_j}(\mathbf{s}^{(j)}, \ell - 2n_j)$. Let $W_j(\lambda)$ be the matrix

$$W_j(\lambda) = \begin{pmatrix} 0 & -\frac{\varepsilon_j}{a_j} \\ \varepsilon_j a_j & \frac{p_j(\lambda)}{a_j} \end{pmatrix}, \quad j \in \mathbb{N}. \tag{4.13}$$

associated with the transformation $\mathcal{T}_j$ $(1 \le j \le N)$.

After the j-th step we obtain the following representation for the solution φ of the moment problem $MP_\kappa(\mathbf{s}, \ell)$

$$\begin{aligned} \varphi(\lambda) &= \mathcal{T}_1 \circ \cdots \circ \mathcal{T}_j[\varphi_j(\lambda)] \\ &= -\frac{\varepsilon_1}{p_1(\lambda)} - \frac{\varepsilon_1\varepsilon_2 a_1^2}{p_2(\lambda)} - \cdots - \frac{\varepsilon_{j-1}\varepsilon_j a_{j-1}^2}{p_j(\lambda) + \varepsilon_j a_j^2 \varphi_j(\lambda)}, \end{aligned} \tag{4.14}$$

where the last formula stands for the continuous fraction expansion (this shorthand notation is often used in the literature). The resulting matrix $W_{[1,j]}(\lambda)$ of the linear fractional transformation in (4.14) coincides with the product of the matrices W_i $(1 \le i \le j)$

$$W_{[1,j]}(\lambda) = W_1(\lambda) \dots W_j(\lambda) \quad (j \le N). \tag{4.15}$$

Theorem 4.4. *Let $n_1 < \cdots < n_N (\le n)$ be a sequence of all normal indices of S_n and let the matrix $W_{[1,j]}(\lambda) = \begin{pmatrix} w_{11}^{(j)}(\lambda) & w_{12}^{(j)}(\lambda) \\ w_{21}^{(j)}(\lambda) & w_{22}^{(j)}(\lambda) \end{pmatrix}$ be given by (4.15). Then for every $j \le N-1$ the formula*

$$\varphi(\lambda) = \mathcal{T}_{W_{[1,j]}(\lambda)}[\varphi_j(\lambda)] := \frac{w_{11}^{(j)}(\lambda)\varphi_j(\lambda) + w_{12}^{(j)}(\lambda)}{w_{21}^{(j)}(\lambda)\varphi_j(\lambda) + w_{22}^{(j)}(\lambda)}, \tag{4.16}$$

establishes a one-to-one correspondence between the sets $\mathcal{I}_\kappa(\mathbf{s}, \ell)$ and $\mathcal{I}_{\kappa-\kappa_j}(\mathbf{s}^{(j)}, \ell - 2n_j)$, where $\mathbf{s}^{(j)}$ is defined recursively by (4.7) and $\kappa_j = \nu_-(S_{n_j-1})$. Moreover,

$$\varphi \in \mathcal{M}_\kappa(\mathbf{s}, \ell) \Leftrightarrow \varphi_j \in \mathcal{M}_{\kappa-\kappa_j}(\mathbf{s}^{(j)}, \ell - 2n_j)$$

In the case, when $\det S_n = 0$ *the statement remains valid for $j = N$.*

Proof. The proof is obtained by successive application of the Schur algorithm described above and Propositions 4.1, 4.2, and Corollary 4.3 to the problem $MP_\kappa(\mathbf{s}, \ell)$. In the nondegenerate case this process terminates when $j = N-1$, since $n < n_N$. In the degenerate case Propositions 4.1 can be applied one more time, since $n \ge n_N$. $\square$

To find an explicit form of the matrix $W_{[1,j]}(\lambda)$ let us define the so-called polynomials $P_j(\lambda)$ and $Q_j(\lambda)$ of the first and the second kind, respectively, as solutions of the difference equation

$$\varepsilon_{j-1}\varepsilon_j a_{j-1}u_{j-2} - p_j(\lambda)u_{j-1} + a_j u_j = 0 \ \ (j = \overline{1,N}), \tag{4.17}$$

with the initial conditions

$$\begin{aligned} P_0(\lambda) = 1, \quad P_1(\lambda) &= \frac{p_1(\lambda)}{a_1}, \\ Q_0(\lambda) = 0, \quad Q_1(\lambda) &= \frac{\varepsilon_1}{a_1}. \end{aligned} \tag{4.18}$$

As is easily seen from (4.17)

$$\deg P_j = \sum_{i=1}^{j} n_i, \ \deg Q_j = \sum_{i=1}^{j-1} n_i \quad (j \geq 1).$$

Theorem 4.5. *The resolvent matrix $W_{[1,j]}(\lambda)$ in (4.15) admits the following representation*

$$W_{[1,j]}(\lambda) = \begin{pmatrix} -\varepsilon_j a_j Q_{j-1}(\lambda) & -Q_j(\lambda) \\ \varepsilon_j a_j P_{j-1}(\lambda) & P_j(\lambda) \end{pmatrix}, \tag{4.19}$$

where P_j and Q_j $(1 \leq j \leq N)$ are polynomials of the first and the second kind associated with the matrix S_n via (4.17), (4.18).

Proof. For $j = 1$ the formula (4.19) coincides with (4.13). Proceed by induction and assume that (4.19) holds for $j := j-1$. Then it follows from (4.19), (4.13) and the difference equation (4.17) that

$$\begin{aligned} W_{[1,j]}(\lambda)\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= W_{[1,j-1]}(\lambda)W_j(\lambda)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -\varepsilon_{j-1}a_{j-1}Q_{j-2}(\lambda) & -Q_{j-1}(\lambda) \\ \varepsilon_{j-1}a_{j-1}P_{j-2}(\lambda) & P_{j-1}(\lambda) \end{pmatrix}\begin{pmatrix} -\varepsilon_j/a_j \\ p_j(\lambda)/a_j \end{pmatrix} \\ &= \frac{1}{a_j}\begin{pmatrix} \varepsilon_{j-1}\varepsilon_j a_{j-1}Q_{j-2}(\lambda) - p_j(\lambda)Q_{j-1}(\lambda) \\ -\varepsilon_{j-1}\varepsilon_j a_{j-1}P_{j-2}(\lambda) + p_j P_{j-1}(\lambda) \end{pmatrix}. \end{aligned}$$

Due to the difference equation (4.17)

$$W_{[1,j]}(\lambda)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -Q_j(\lambda) \\ P_j(\lambda) \end{pmatrix}. \tag{4.20}$$

Hence one obtains

$$W_{[1,j]}(\lambda)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = W_{[1,j-1]}(\lambda)\begin{pmatrix} 0 \\ \varepsilon_j a_j \end{pmatrix} = \varepsilon_j a_j \begin{pmatrix} -Q_{j-1}(\lambda) \\ P_{j-1}(\lambda) \end{pmatrix}. \tag{4.21}$$

The formulas (4.20)-(4.21) prove (4.19). □

Remark 4.6. The recursion algorithm for Nevanlinna functions is well known (see for example [1]). The formula (4.19) for the resolvent matrix can be found in [23], were truncated moment problems were studied. The operator approach to such problems was presented in [15], [16]. In the indefinite case this algorithm was studied by M. Derevyagin in [9], formulas (4.19) for the matrix $W(\lambda)$ and the statement of the theorem for the nondegenerate even moment problem were proven in [10]. The linear fractional transformations similar to $\mathcal{T}_j$ (so-called Schur transform) has been studied by D. Alpay, A. Dijksma and H. Langer in [3], [4].

5. Description of solutions

In this section we find a solvability criterion and describe the set of solutions of the problems $MP_\kappa(\mathbf{s},\ell)$ and $IP_\kappa(\mathbf{s},\ell)$ in the general setting. As well as in the case of basic problem we will distinguish non-degenerate problems and two types of degenerate problems:

(A) $\operatorname{rank} S_n = n_N = \operatorname{rank} S_{n_N-1}$;
(B) $\operatorname{rank} S_n > n_N = \operatorname{rank} S_{n_N-1}$.

5.1. Non-degenerate moment problem.

Theorem 5.1. *Let $(\mathbf{s},\ell)$ be a sequence of real numbers such that $\det S_n \neq 0$, Then the moment problem $MP_\kappa(\mathbf{s},\ell)$ is solvable if and only if*

$$\kappa \geq \nu_-.$$

The sets $\mathcal{M}_\kappa(\mathbf{s},\ell)$ and $\mathcal{I}_\kappa(\mathbf{s},\ell)$ are parametrized by the formula

$$\varphi(\lambda) = \frac{w_{11}^{(N)}(\lambda)\tau(\lambda) + w_{12}^{(N)}(\lambda)}{w_{21}^{(N)}(\lambda)\tau(\lambda) + w_{22}^{(N)}(\lambda)}, \tag{5.1}$$

where in the even case

$$\varphi \in \mathcal{M}_\kappa(\mathbf{s},\ell) \Leftrightarrow \tau \in \mathbf{N}_{\kappa-\nu_-} \text{ and satisfies (E)};$$

and in the odd case

$$\varphi \in \mathcal{I}_\kappa(\mathbf{s},\ell) \Leftrightarrow \tau \in \mathbf{N}_{\kappa-\nu_-} \text{ and satisfies (O)};$$
$$\varphi \in \mathcal{M}_\kappa(\mathbf{s},\ell) \Leftrightarrow \tau \in \mathbf{N}_{\kappa-\nu_-,1} \text{ and satisfies (O)}.$$

Proof. The proof is obtained by compilation of Theorem 4.4, Theorem 4.5 and Proposition 3.1. □

This result seems to be new even for the odd Hamburger moment problem.

Corollary 5.2. *Let $s_0, s_1, \dots, s_{2n+1}$ be real numbers, such that $S_n > 0$ and $\det S_n \neq 0$. Then the moment problem* (1.1) *with $\ell = 2n+1$ is solvable and the formula* (1.5) *describes the set of solutions of* (1.1) *when τ is ranging over the class $\mathbf{N}_{0,1}$ and satisfies the condition (O). Moreover, $\varphi \in \mathcal{I}_0(\mathbf{s}, 2n+1)$ if and only if τ belongs to $\mathbf{N}_0$ and satisfies (O).*

The following example shows the importance of the condition $\tau \in \mathbf{N}_{0,1}$ in Corollary 5.2.

Example. Let $s_0 = 1$,$s_1 = 0$. Then $p(\lambda) = \lambda$ and the set of solutions of the problem $\mathcal{I}_0(\mathbf{s},1)$ is described by

$$\varphi(\lambda) = \frac{-1}{p(\lambda) + \tau(\lambda)},$$

where $\tau \in \mathbf{N}_0$ and $\tau(\lambda) = o(1)$ as $\lambda \hat{\to} \infty$. The function $\tau(\lambda) = -\frac{1}{i\ln(1+\sqrt{\lambda})}$ belongs to the class $\mathbf{N}_0$ and satisfies the condition $\tau(\lambda) = o(1)$ as $\lambda \hat{\to} \infty$. Therefore,

$$\varphi(\lambda) = \frac{-i\ln(1+\sqrt{\lambda})}{i\lambda\ln(1+\sqrt{\lambda}) - 1}$$

is a solution of the problem $IP_0(\mathbf{s},1)$

$$\varphi(\lambda) = \frac{-1}{\lambda} + o\left(\frac{1}{\lambda^2}\right) \text{ as } \lambda \hat{\to} \infty.$$

However, $\tau \notin \mathbf{N}_{0,1}$ (see [20]) and, hence, φ is not a solution of the moment problem (1.1).

5.2. **Degenerate moment problem. Case (A).** In Theorem 5.8 solvability criteria for degenerate moment problems with minimal negative signature $\kappa = \nu_-(S_n)$ are given. We start with two lemmas.

Lemma 5.3. *Let* $(\mathbf{s}, 2n)$ *be a sequence of real numbers such that* $\det S_n = 0$. *Then the Hankel rank* n_N *of the sequence* $(\mathbf{s}, 2n)$ *coincides with the largest normal index* n_N *of the Hankel matrix* S_n.

Proof. By Frobenius Theorem (see [19, Lemma X.10.1]), if r is the smallest integer r $(0 \le r \le n)$, such that (1.7) holds, then $\det S_{r-1} \neq 0$. Hence r is the normal index of S_n. Moreover, r is the largest normal index of S_n, since the vectors $(s_j, \dots, s_{j+n})^\top$, $(0 \le j \le n_N)$ in (1.7) are linearly dependent. This implies that $\det S_n = 0$ for all $j \ge r$. □

Lemma 5.4. *Let* $(\mathbf{s}, 2n)$ *be a sequence of real numbers such that* $\det S_n = 0$, *let* n_N *be the largest normal index of the Hankel matrix* S_n *and let* S_n *admit a Hankel extension* S_{n+1}, *such that* $\nu_-(S_{n+1}) = \nu_-(S_n)$. *Then there are real numbers* $\alpha_0, \dots, \alpha_{n_N-1}$, *such that*

$$s_j = \alpha_0 s_{j-n_N} + \dots + \alpha_{n_N-1} s_{j-1} \quad (n_N \le j \le 2n+1); \tag{5.2}$$

$$s_{2n} \ge \alpha_0 s_{2n-n_N+2} + \dots + \alpha_{n_N-1} s_{2n+1}. \tag{5.3}$$

Proof. Let us set $v_j = (s_j, \dots, s_{j+n})^\top$, $(0 \le j \le n+1)$. Since $\nu_-(S_n) = \nu_-(S_{n+1})$, it follows from Lemma A.2 that

$$v_{n+1} \in \text{span } (v_0, \dots, v_n),$$

i.e. there is $c \in \mathbb{C}^{n+1}$ such that

$$v_{n+1} = S_n c. \tag{5.4}$$

By Lemma 5.3 there are real numbers $\alpha_0, \dots, \alpha_{r-1}$, such that

$$v_r = \begin{pmatrix} s_r \\ \vdots \\ s_{r+n} \end{pmatrix} = \begin{pmatrix} s_0 & \dots & s_n \\ \vdots & \iddots & \vdots \\ s_n & \dots & s_{2n} \end{pmatrix} \widehat{\alpha}, \quad \text{where } \widehat{\alpha} := \begin{pmatrix} \alpha_0 \\ \dots \\ \alpha_{r-1} \\ 0_{(n-r+1)\times 1} \end{pmatrix} \tag{5.5}$$

This together with (5.4) implies, in particular, that

$$s_{r+n+1} = c^* v_r = c^* S_n \widehat{\alpha} = v_{n+1}^* \widehat{\alpha} = \begin{pmatrix} s_{n+1} & \dots & s_{2n+1} \end{pmatrix} \widehat{\alpha}. \tag{5.6}$$

Denote by V the $(n+1)\times(n+1)$ forward shift matrix $V = (\delta_{i,j+1})_{i,j=1}^{n+1}$. Then (5.2) and (5.6) imply

$$v_{r+1} = \begin{pmatrix} s_{r+1} \\ \vdots \\ s_{r+n+1} \end{pmatrix} = \begin{pmatrix} s_1 & \dots & s_{n+1} \\ \vdots & \iddots & \vdots \\ s_{n+1} & \dots & s_{2n+1} \end{pmatrix} \widehat{\alpha} = S_n V \widehat{\alpha}$$

Iterating these calculations one obtains for $(0 \le) j \le n$

$$v_{j+1} = \begin{pmatrix} s_{j+1} \\ \vdots \\ s_{j+n+1} \end{pmatrix} = \begin{pmatrix} s_1 & \dots & s_{n+1} \\ \vdots & \iddots & \vdots \\ s_{n+1} & \dots & s_{2n+1} \end{pmatrix} V^{j-r} \widehat{\alpha} = S_n V^{j-r+1} \widehat{\alpha}, \tag{5.7}$$

which proves (5.2). Setting in (5.7) $j = n$ one obtains

$$v_{n+1} = S_n V^{n-r+1}\widehat{\alpha}.$$

Then it follows from Lemma A.2 that

$$\begin{aligned} s_{2n+2} &\geq (V^{n-r+1}\widehat{\alpha})^* S_n V^{n-r+1}\widehat{\alpha} \\ &= (V^{n-r+1}\widehat{\alpha})^* \begin{pmatrix} s_{n+1} \\ \vdots \\ s_{2n+1} \end{pmatrix} \\ &= \alpha_0 s_{2n-r+2} + \cdots + \alpha_{r-1} s_{2n+1}. \end{aligned}$$

Hence (5.3) holds and this completes the proof. □

This motivates the following definition which in the definite case was used in [8].

Definition 5.5. A sequence $(\mathbf{s}, \ell) = \{s_j\}_{j=0}^{\ell}$ with the Hankel rank $r = \operatorname{rank} \mathbf{s}$ is called *recursively generated*, if there exist numbers $\alpha_0, \dots, \alpha_{r-1}$, such that

$$s_j = \alpha_0 s_{j-r} + \cdots + \alpha_{r-1} s_{j-1} \quad (r \leq j \leq \ell). \tag{5.8}$$

Theorem 5.6. *Let $(\mathbf{s}, \ell)$ be a sequence of real numbers such that $\det S_n = 0$, $n = [\ell/2]$, let $n_1 < \cdots < n_N$ be all normal indices of the degenerate Hankel matrix S_n, let $(\mathbf{s}^{(N)}, \ell - 2n_N)$ be a sequence of induced moments determined by successive application of (4.7), and let $\kappa = \nu_-(S_n)$, $\kappa_N = \nu_-(S_{n_N-1})$. In the case when $\ell = 2n$ is even the following statements are equivalent:*

- (i) *The moment problem $\mathcal{M}_\kappa(\mathbf{s}, \ell)$ is solvable;*
- (ii) *The moment problem $\mathcal{M}_0(\mathbf{s}^{(N)}, \ell - 2n_N)$ is solvable;*
- (iii) $s_0^{(N)} = \cdots = s_{\ell-2n_N}^{(N)} = 0$;
- (iv) *S_n admits a Hankel extension S_{n+1} such that $\nu_-(S_{n+1}) = \nu_-(S_n)$;*
- (v) *$(\mathbf{s}, \ell)$ is recursively generated;*
- (vi) *rank $S_n = n_N$;*

If $\ell = 2n + 1$ is odd, then

$$(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv') \Leftrightarrow (v) \Leftrightarrow (vi)$$

where (iv') and (vi') take the form:

- (iv') *there exists a real number s_{2n+2}, such that $\nu_-(S_{n+1}) = \nu_-(S_n)$;*
- (vi') *rank $S_n = n_N$ and $s_{\ell-2n_N}^{(N)} = 0$.*

If one of the above conditions holds, then $\kappa = \kappa_N$ and the moment problem $\mathcal{M}_\kappa(\mathbf{s}, \ell)$ has the unique solution given by

$$\varphi(\lambda) = -\frac{Q_N(\lambda)}{P_N(\lambda)}. \tag{5.9}$$

Proof. **Even case.** (i) $\Rightarrow$ (iii) If the moment problem $MP_\kappa(\mathbf{s}, 2n)$ is solvable then by Theorem 4.4 the problem $MP_{\kappa-\kappa_N}(\mathbf{s}^{(N)}, 2(n - n_N))$ is also solvable. If (iii) is not in force then it follows from Proposition 3.3 that

$$\kappa - \kappa_N \geq \nu_-(S_{n-n_N}^{(N)}) + \nu_0(S_{n-n_N}^{(N)}).$$

In view of Corollary 4.3

$$\nu_-(S_{n-n_N}^{(N)}) = \nu_-(S_n) - \nu_-(S_{n_N-1}) = \kappa - \kappa_N. \tag{5.10}$$

Therefore,

$$\nu_-(S^{(N)}_{n-n_N}) \geq \nu_-(S^{(N)}_{n-n_N}) + \nu_0(S^{(N)}_{n-n_N}),$$

which implies $\nu_0(S^{(N)}_{n-n_N}) = 0$. But by the same Corollary 4.3 $\nu_0(S^{(N)}_{n-n_N}) = \nu_0(S_n) \neq 0$.

(iii) $\Rightarrow$ (ii) If (iii) holds then $\nu_-(S^{(N)}_{n-n_N}) = 0$ and by Theorem 1.1 $\varphi_N(\lambda) \equiv 0$ is the unique solution of the problem $MP_0(\mathbf{s}^{(N)}, 2(n-n_N))$. The equality $\kappa = \kappa_N$ is implied by (5.10).

(ii) $\Rightarrow$ (i) This follows from Theorem 4.4.

(iii) $\Rightarrow$ (iv) If (iii) holds then in view of Theorem 4.4 and Theorem 4.5 the unique solution of the problem $MP_\kappa(\mathbf{s}, 2n)$ is given by

$$\varphi(\lambda) = -\frac{Q_N(\lambda)}{P_N(\lambda)} \in \mathbf{N}_\kappa.$$

Since φ is rational of rank $\varphi = n_N$ it admits the asymptotic expansion (1.3) for every n, in particular, there exist s_{2n+1}, $s_{2n+2} \in \mathbb{R}$ such that

$$\varphi(\lambda) = -\frac{s_0}{\lambda} - \frac{s_1}{\lambda^2} - \cdots - \frac{s_{2n+2}}{\lambda^{2n+3}} + o\left(\frac{1}{\lambda^{2n+3}}\right) \quad (\lambda \widehat{\to} \infty)$$

and the corresponding Hankel matrix S_{n+1} has the same rank as S_n. This implies that $\nu_-(S_{n+1}) = \nu_-(S_n)$.

(iv) $\Rightarrow$ (v) Assume that S_n admits a Hankel extension S_{n+1} such that $\nu_-(S_{n+1}) = \nu_-(S_n)$. Then by Lemma 5.3 there are $\alpha_0, \dots, \alpha_{n_N-1}$, such that

$$s_j = \alpha_0 s_{j-n_N} + \cdots + \alpha_{n_N-1} s_{j-1} \quad (n_N \leq j \leq 2n+1). \tag{5.11}$$

By Definition 5.5 this means that $(\mathbf{s}, 2n)$ is recursively generated.

(v) $\Rightarrow$ (vi) The condition (v) implies that

$$\begin{pmatrix} s_j \\ \vdots \\ s_{j+n} \end{pmatrix} \in \operatorname{span} \left\{ \begin{pmatrix} s_0 \\ \vdots \\ s_n \end{pmatrix}, \dots \begin{pmatrix} s_{n_N-1} \\ \vdots \\ s_{n_N-1+n} \end{pmatrix} \right\} \tag{5.12}$$

for all $j \leq n$, and therefore, rank $S_n = n_N$.

(vi) $\Rightarrow$ (iii) If rank $S_n = n_N$ then $\nu_0(S_n) = n - n_N + 1$. By Lemma A.3

$$\nu_0(S^{(N)}_{n-n_N}) = \nu_0(S_n) = n - n_N + 1$$

and, hence, $s_j^{(N)} = 0$ for all j such that $0 \leq j \leq 2(n-n_N)$.

Odd case. In the odd case the proof of the equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv')$ is pretty much the same.

(iv') $\Rightarrow$ (v). Assume that there exists s_{2n+2} such that $\nu_-(S_{n+1}) = \nu_-(S_n)$. Then by Lemma 5.3 there are $\alpha_0, \dots, \alpha_{n_N-1}$, such that (5.11) holds for all for all $(n_N \leq) j \leq 2n+1$. This implies, that the sequence $(\mathbf{s}, 2n+1)$ is recursively generated.

(v) $\Rightarrow$ (vi') The statement (v) implies that (5.12) holds for all $j \leq n+1$ and hence there exist $\beta_1, \dots, \beta_{n_N}$, such that

$$\begin{pmatrix} s_{n+1} \\ \vdots \\ s_{2n+1} \end{pmatrix} = \beta_1 \begin{pmatrix} s_0 \\ \vdots \\ s_n \end{pmatrix} + \cdots + \beta_{n_N} \begin{pmatrix} s_{n_N-1} \\ \vdots \\ s_{n_N-1+n} \end{pmatrix}.$$

Therefore, rank $S_n = n_N$. Let us set

$$s_{2n+2} = \beta_1 s_{n+1} + \cdots + \beta_{n_N} s_{n+n_N}.$$

Then rank $S_{n+1} =$ rank $S_n = n_N$ and by (iii) we obtain

$$s_0^{(N)} = \cdots = s_{2n+1-2n_N}^{(N)} = s_{2n+2-2n_N}^{(N)} = 0.$$

(vi') $\Rightarrow$ (iii) If rank $S_n = n_N$ then it was shown above that $s_j^{(N)} = 0$ for all $j \le 2(n - n_N)$. Now (iii) holds since $s_{2n+1-2n_N}^{(N)} = 0$. □

Remark 5.7. If φ is a rational function of degree r and φ has the asymptotic expansion (1.3), then for $n \ge r-1$, by Kronecker theorem rank $S_n = r$ and $\varphi \in \mathbf{N}_\kappa$, where $\kappa = \nu_-(S_n)$ (see [19, Theorem 16.11.9]). Then by Theorem 5.6 the problem $IP_\kappa(\mathbf{s}, 2n)$ has a unique solution.

Now, let $\psi \in \mathbf{N}_\kappa$ be such that

$$\psi(\lambda) = \varphi(\lambda) + o\left(\frac{1}{\lambda^{2r+1}}\right) \text{ as } \lambda \widehat{\rightarrow} \infty.$$

Then both φ and ψ are solutions of the problem $IP_\kappa(\mathbf{s}, 2n)$ and hence $\psi(\lambda) \equiv \varphi(\lambda)$. This proves the rigidity result for generalized Nevanlinna functions obtained in [5] and proved originally by Burns and Krantz for functions from the Schur class, [7].

In the next theorem we describe solutions of the problems $MP_\kappa(\mathbf{s}, \ell)$ and $MI_\kappa(\mathbf{s}, \ell)$ in the case where the rank of the Hankel matrix S_n coincides with the Hankel rank of the sequence.

Theorem 5.8. *Let* $(\mathbf{s}, \ell)$ *be a sequence of real numbers such that* $\det S_n = 0$ *(*$n = [\ell/2]$*) and let*

$$\operatorname{rank} S_n = n_N.$$

Then the problems $MP_\kappa(\mathbf{s}, \ell)$ *and* $MI_\kappa(\mathbf{s}, \ell)$ *are solvable if and only if:*

(i) *either* $\kappa = \nu_-(S_n)$ *and the equivalent conditions of Theorem 5.6 are satisfied;*

(ii) *or* $\kappa \ge \nu_-(S_n) + \nu_0(S_n)$.

If $\kappa \ge \nu_-(S_n) + \nu_0(S_n)$, $W_{[1,N]}(\lambda)$ *is given by* (4.19)*, and*

$$W(\lambda) = W_{[1,N]}(\lambda) \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{2\nu_0} \end{pmatrix} \tag{5.13}$$

then the formula

$$\varphi(\lambda) = \mathcal{T}_{W(\lambda)}[\tau(\lambda)], \tag{5.14}$$

describes the sets $\mathcal{M}_\kappa(\mathbf{s}, \ell)$ *and* $\mathcal{I}_\kappa(\mathbf{s}, \ell)$ *as follows: in the even case*

$$\varphi \in \mathcal{M}_\kappa(\mathbf{s}, \ell) \Leftrightarrow \tau \in \mathbf{N}_{\kappa-\nu} \textit{ and satisfies (E)};$$

and in the odd case

$$\varphi \in \mathcal{I}_\kappa(\mathbf{s}, \ell) \Leftrightarrow \tau + s_{\ell-2n_N}^{(N)} \in \mathbf{N}_{\kappa-\nu} \textit{ and satisfies (O)};$$

$$\varphi \in \mathcal{M}_\kappa(\mathbf{s}, \ell) \Leftrightarrow \tau + s_{\ell-2n_N}^{(N)} \in \mathbf{N}_{\kappa-\nu,1} \textit{ and satisfies (O)}.$$

Proof. It follows from Theorem 4.4 that the problem $MP_\kappa(\mathbf{s},\ell)$ is solvable if and only if the basic problem $MP_{\kappa-\nu_-(S_{n_N-1})}(\mathbf{s}^{(N)},\ell-2n_N)$ is solvable. Since $\operatorname{rank} S_n = \operatorname{rank} S_{n_N-1}$, then $\nu_\pm(S_n)=\nu_\pm(S_{n_N-1})$, and one obtains from Corollary 4.3 that $\operatorname{rank} S^{(N)}_{n-n_N}=0$. Hence the basic problem $\mathcal{M}_{\kappa-\nu_-}(\mathbf{s}^{(N)},2(n-n_N))$ is of type (A). The rest of the statements are immediate from Lemma 3.2. □

5.3. **Degenerate moment problem. Case (B).** In this subsection we give solvability criteria and describe solutions of the problems $MP_\kappa(\mathbf{s},\ell)$ and $MI_\kappa(\mathbf{s},\ell)$ in the case where the rank of the Hankel matrix S_n is greater then the Hankel rank of the sequence $(\mathbf{s},\ell)$.

Theorem 5.9. *Let* $(\mathbf{s},\ell)$ *be a sequence of real numbers such that* $\det S_n=0$ *(*$n=[\ell/2]$*) and let*

$$\operatorname{rank} S_n > n_N,$$

Then the moment problem $\mathcal{M}_\kappa(\mathbf{s},\ell)$ *is solvable if and only if* (1.13) *holds.*

Let $\nu_-:=\nu_-(S_n)$, $\nu_0:=\nu_0(S_n)$, *let* ν *be defined by* (2.17), *let* $(\mathbf{s}^{(N)},\ell-2n_N)$ *be the sequence of induced moments after* N *steps, let the integer* m *be defined by*

$$s_j^{(N)}=0 \text{ for } j<m; \quad \varepsilon:=s_m^{(N)}\neq 0,$$

let the polynomial $\widehat{p}$ *be defined by the formulas* (3.13) *and* (3.15), *where* $n:=n-n_N$, *let the matrix-valued function* $W_{[1,N]}(\lambda)$ *be given by* (4.19), *and finally let*

$$W(\lambda)=W_{[1,N]}(\lambda)\begin{pmatrix} 0 & -\widehat{\varepsilon} \\ \lambda^{2\nu_0}\widehat{\varepsilon} & \lambda^{2\nu_0}\widehat{p}(\lambda)\end{pmatrix}$$

If κ *satisfies* (1.13), *then the formula* (5.14), *describes the sets* $\mathcal{M}_\kappa(\mathbf{s},\ell)$ *and* $\mathcal{I}_\kappa(\mathbf{s},\ell)$ *as follows: in the even case*

$$\varphi\in\mathcal{M}_\kappa(\mathbf{s},\ell)\Leftrightarrow\tau\in\mathbf{N}_{\kappa-\nu-\nu_-} \text{ and satisfies (E)}\ .$$

and in the odd case

$$\varphi\in\mathcal{I}_\kappa(\mathbf{s},\ell)\Leftrightarrow\tau+s^{(N)}_{\ell-2n_N}\in\mathbf{N}_{\kappa-\nu-\nu_-} \text{ and satisfies (O)}\ ;$$

$$\varphi\in\mathcal{M}_\kappa(\mathbf{s},\ell)\Leftrightarrow\tau+s^{(N)}_{\ell-2n_N}\in\mathbf{N}_{\kappa-\nu-\nu_-,1} \text{ and satisfies (O)}\ .$$

Proof. It follows from Theorem 4.4 that the problem $MP_\kappa(\mathbf{s},\ell)$ is solvable if and only if the basic problem $MP_{\kappa-\nu_-(S_{n_N-1})}(\mathbf{s}^{(N)},\ell-2n_N)$ of type (B) is solvable. By Proposition 3.3 the latter problem is solvable if and only if

$$\kappa-\nu_-(S_{n_N-1})\geq\nu_0(S^{(N)}_{n-n_N})+\nu_-(S^{(N)}_{n-n_N}).$$

Due to Corollary 4.3

$$\nu_0(S^{(N)}_{n-n_N})=\nu_0(S_n),\quad \nu_-(S^{(N)}_{n-n_N})=\nu_-(S_n)-\nu_-(S_{n_N-1}).$$

These proves the criterion (1.13). The statement for the problem $IP_\kappa(\mathbf{s},\ell)$ is proved by the same reasonings.

The second part of the proof is implied by Theorem 4.4, Theorem 4.5 and Theorem 3.4. □

Appendix A. Some lemmas on matrices

A.1. **Some lemmas for block matrices.** The verification of the first lemma is left to the reader.

Lemma A.1. *The inverse of the invertible block matrix $\widetilde{A}$ of the form*

$$\widetilde{A} = \begin{pmatrix} 0 & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{13}^* & A_{23}^* & A_{33} \end{pmatrix}$$

is given by

$$\widetilde{A}^{-1} = \begin{pmatrix} -A_{13}^{-*}(A_{33} - A_{23}^* A_{22}^{-1} A_{23}) A_{13}^{-1} & -A_{13}^{-*} A_{23}^* A_{22}^{-1} & A_{13}^{-*} \\ -A_{22}^{-1} A_{23} A_{13}^{-1} & A_{22}^{-1} & 0 \\ A_{13}^{-1} & 0 & 0 \end{pmatrix}$$

The following lemma extends a well-known result for nonnegative block matrices.

Lemma A.2. *Let $\widetilde{A}$ be a Hermitian matrix of the form*

$$\widetilde{A} = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}, \quad A \in \mathbb{C}^{n\times n}, \ C \in \mathbb{C}^{m\times m}$$

such that $\nu_-(\widetilde{A}) = \nu_-(A)$. Then

(i) $\operatorname{ran} B \subseteq \operatorname{ran} A$*;*

(ii) *if $Bh = Ag$ for some $h \in \mathbb{C}^m$, $g \in \mathbb{C}^n$, then $h^*Ch \geq g^*Ag$.*

Proof. First assume that A is invertible. Then the identity

$$\widetilde{A} = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \begin{pmatrix} I & 0 \\ B^*A^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & C - B^*A^{-1}B \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix},$$

and the assumption $\nu_-(\widetilde{A}) = \nu_-(A)$ shows that $C - B^*A^{-1}B \geq 0$. If A is not invertible, then there is a block decomposition

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} B_0 \\ B_1 \end{pmatrix},$$

where A_0 is invertible. Note that $\nu_-(A) = \nu_-(A_0)$. Now the previous statement can be applied to the block matrix

$$\widetilde{A} = \begin{pmatrix} A_0 & 0 & B_0 \\ 0 & 0 & B_1 \\ B_0^* & B_1^* & C \end{pmatrix}$$

with the invertible matrix A_0 in the left upper corner, since $\nu_-(\widetilde{A}) = \nu_-(A_0)$. Thus, one concludes that

$$\begin{pmatrix} 0 & B_1 \\ B_1^* & C \end{pmatrix} - \begin{pmatrix} 0 \\ B_0^* \end{pmatrix} A_0^{-1} \begin{pmatrix} 0 & B_0 \end{pmatrix} = \begin{pmatrix} 0 & B_1 \\ B_1^* & C - B_0^*A_0^{-1}B_0 \end{pmatrix} \geq 0.$$

This implies $B_1 = 0$ and

$$C - B_0^*A_0^{-1}B_0 \geq 0. \tag{A.1}$$

The identity $B_1 = 0$ means that $\operatorname{ran} B \subseteq \operatorname{ran} A_0 = \operatorname{ran} A$ proving the range inclusion in (i). On the other hand, it follows from (A.1) that the vectors h and g_0 for which $B_0h = A_0g_0$ satisfy the inequality

$$h^*Ch \geq h^*B_0^*A_0^{-1}B_0h = g_0^*A_0g_0 = g^*Ag.$$

Now (ii) is implied by this inequality. $\square$

A.2. **Some results for Hankel matrices.**

Lemma A.3. *Let S_n be a Hankel matrix, let n_1 be the first normal index of S_n, and let the polynomial $p_1(\lambda) = p_{n_1}^{(1)}\lambda^{n_1} + \cdots + p_0^{(1)}$ and the moment sequence $(\mathbf{s}^{(1)}, \ell - 2n_1)$ be defined by (4.7). Then for all $i = \overline{0, n-n_1}$*

$$\nu_\pm(S_i^{(1)}) = \nu_\pm(S_{i+n_1}) - \nu_\pm(S_{n_1-1}); \tag{A.2}$$

$$\nu_0(S_i^{(1)}) = \nu_0(S_{i+n_1}). \tag{A.3}$$

Proof. Consider the equation (4.7) with $j = 2i$. Multiplying it both from the left and from the right by the matrix J_{2i+n_1+2} one obtains the equality

$$\begin{pmatrix} 0 & \widetilde{A}_{12} \\ \widetilde{A}_{12}^* & \widetilde{A}_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix} = \varepsilon_1 I_{2i+n_1+2}, \tag{A.4}$$

where

$$\widetilde{A}_{12} = \begin{pmatrix} 0 & \dots & \dots & \dots & 0 & s_{n_1-1} \\ \vdots & & & \iddots & \iddots & \vdots \\ 0 & \dots & 0 & s_{n_1-1} & \dots & s_{n_1+i-1} \end{pmatrix} \in \mathbb{C}^{(i+1)\times(n_1+i+1)},$$

$$\widetilde{A}_{22} = S_{n_1+i} \in \mathbb{C}^{(n_1+i+1)\times(n_1+i+1)},$$

$$B_{11} = -\varepsilon_1 a_1^2 J_{i+1} S_i^{(1)} J_{i+1} \in \mathbb{C}^{(i+1)\times(i+1)} \tag{A.5}$$

and B_{12}, B_{22} are some matrices from $\mathbb{C}^{(i+1)\times(i+n_1+1)}$ and $\mathbb{C}^{(i+n_1+1)\times(i+n_1+1)}$, respectively. Let us decompose the matrices $\widetilde{A}_{12}$, $\widetilde{A}_{22}$ as follows

$$\widetilde{A}_{12} = \begin{pmatrix} 0_{(i+1)\times n_1} & A_{13} \end{pmatrix}, \qquad A_{13} = \begin{pmatrix} & & s_{n_1-1} \\ & \iddots & \vdots \\ s_{n_1-1} & \dots & s_{n_1+i-1} \end{pmatrix} \in \mathbb{C}^{(i+1)\times(i+1)},$$

$$\widetilde{A}_{22} = \begin{pmatrix} A_{22} & A_{23} \\ A_{23}^* & A_{33} \end{pmatrix} = \begin{pmatrix} S_{n_1-1} & A_{23} \\ A_{23}^* & A_{33} \end{pmatrix}, \qquad \begin{matrix} A_{23} \in \mathbb{C}^{n_1\times(i+1)}, \\ A_{33} \in \mathbb{C}^{(i+1)\times(i+1)}. \end{matrix}$$

Since the matrices $A_{13} = A_{13}^*$, $A_{22} = S_{n_1-1}$ are invertible, it follows from Lemma A1 that the matrix

$$\widetilde{A} = \begin{pmatrix} 0 & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{13} & A_{23}^* & A_{33} \end{pmatrix}$$

is invertible, and the left upper corner B_{11} of the inverse matrix $\widetilde{A}^{-1}$ is given by

$$\widetilde{B}_{11} = (A^{-1})_{11} = -A_{13}^{-1}(A_{33} - A_{23}^* A_{22}^{-1} A_{23}) A_{13}^{-1} \tag{A.6}$$

It follows from (A.4)-(A.6) that

$$a_1^2 J_{i+1} S_i^{(1)} J_{i+1} = -\varepsilon_1 B_{11} = -(\widetilde{A}^{-1})_{11} = A_{13}^{-1}(A_{33} - A_{23}^* A_{22}^{-1} A_{23}) A_{13}^{-1} \tag{A.7}$$

and hence

$$\nu_\pm(S_i^{(1)}) = \nu_\pm(A_{33} - A_{23}^* A_{22}^{-1} A_{23}); \tag{A.8}$$

$$\nu_0(S_i^{(1)}) = \nu_0(A_{33} - A_{23}^* A_{22}^{-1} A_{23}). \tag{A.9}$$

The numbers of positive (negative, zero) eigenvalues of the Schur complement $A_{33} - A_{23}^* A_{22}^{-1} A_{23}$ can be calculated by the formulas

$$\nu_\pm(A_{33} - A_{23}^* A_{22}^{-1} A_{23}) = \nu_\pm(\widetilde{A}_{22}) - \nu_\pm(A_{22}); \tag{A.10}$$

$$\nu_0(A_{33} - A_{23}^* A_{22}^{-1} A_{23}) = \nu_0(\widetilde{A}_{22}) - \nu_0(A_{22}). \tag{A.11}$$

Since $\widetilde{A}_{22} = S_{i+n_1}$ and $A_{22} = S_{n_1-1}$ is invertible the statements of the Lemma A.3 are implied by (A.8)-(A.11). □

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